PROFESSOR: The *Phase Lines* mathlet helps us to understand the behavior of nonlinear autonomous equations. The large graphing window shows a direction field. The horizontal axis is t, and the vertical axis is y. As you see, the direction field is independent of t. This is the significance of an autonomous equation.

The applet opens with the equation 1 minus y times y minus a, but I want to talk to you about a different equation, namely this one, which can be written as the quantity a minus y, times y, minus 1/4. This is a logistic equation with harvesting.

Specifically, this is a model of the oryx population in a certain game preserve in Kenya, measured in kilooryx and years. The Kenyan government wants to sell permits to kill 1/4 kilooryx per year, and it wants to know how large a game preserve it should create to guarantee a stable oryx population.

The size of the preserve determines what would be the limiting population of oryx which is the number a. We can use this applet to explore the options to recommend to the Kenyan government.

We can control the parameter a using this slider down here. Right now, it's set to a equals 0, so the equation is y prime equals minus 1/4 minus y squared, always negative, so all the solutions decrease. With a game preserve of zero area, the population is guaranteed to collapse. We can draw some solution curves by clicking on this window. You'll notice that any horizontal translate of the solution is another solution to this differential equation. This is another consequence of the fact that it's an autonomous differential equation.

We can increase the value of a using this slider. And when I do, notice that the change in the direction field-- it starts to flatten out, especially in an area just above the t-axis. And when a takes on the value 1, a solution appears, a constant solution in blue. This is an equilibrium solution. We can measure the value of y along it. There's a readout below the screen. It seems that y is about 1/2.

And if we take a equals 1 and y equals 1/2, and plug those values into this equation, you will get y prime equals 0. In other words, you get a constant solution.

Now notice this parabola in the lower left of the screen. As I change a, it moves. That parabola

is the graph of y prime as a function of y. So it depends upon a. When a moves up to the value a equals 1, the parabola moves up and touches the y-axis at the value y equals 1/2.

Now this blue curve represents a steady-state solution, but the Kenyan government would be ill-advised to use a preserve only this big. Because random fluctuations will occasionally drive the population of oryx below y equals 1/2, and as soon as that happens, disaster ensues. The population is guaranteed to collapse.

So let's take a bigger game preserve, make a larger, and when you do that, the parabola moves up, the blue double root splits into two roots, and the equilibrium solution bifurcates into two equilibrium solutions.

The red equilibrium is unstable. Solutions near to it diverge from it. In fact, solutions below it approach infinity. These curves become asymptotic to vertical straight lines. They blow up in finite time. Solutions between the two equilibria move from the lower equilibrium up to the green upper equilibrium, and solutions above the green equilibrium decay towards it. The green solution is a stable equilibrium.

You can see this behavior by looking at the parabola as well. If the value of y is less than the red zero, the value of the parabola is negative. Solutions are falling. Between the two roots, the value of y prime is positive. Solutions are growing. And to the right of this equilibrium, the value of y prime is negative. Solutions are falling again.

You can see all this very neatly using the phase line, which I can invoke using this key here. The phase line carries all the information present in the direction field, but in a much more compact form. You can see the two equilibria marked, and below this equilibrium, all the solutions are decaying. Between the two, the solutions are all increasing, and above the upper equilibrium, the solutions are all decaying. These are indicated by these yellow arrows.

Now the Kenyan government has some leeway. It may predict an accidental variation of, say, 1/2 kilooryx. In that case, it should arrange a game preserve large enough so that the distance between the two critical points is at least 1/2. Then, assuming that the initial population is close to this equilibrium, a deviation of 1/2 won't be a disaster, because the population will recover and return to the stable equilibrium.

Well, we've come a long way. We're now looking at a family of differential equations, when indexed by the parameter a. Each value of a has its own direction field, its own set of critical

points, it's own solutions, and its own phase line. But as we see, for practical policy reasons, it's important to be able to consider them all simultaneously. The bifurcation diagram lets you do exactly that. Let's invoke it with this check box.

The diagram has appeared above the a slider. In it is marked a curve. This curve consists of all the critical points for these differential equations for all values of a, marked simultaneously. The critical points for a given value of a appear above that value of a in the slider. So here's the phase line for this value of a containing an unstable critical point and a stable critical point.

And when I change the value of a, that phase line changes and moves. And as the value of a decreases to a equals 1, those two critical points collide and form a single semi-stable critical point. For smaller values of a, there are no critical points, until you reach the value a equals minus 1, when a single semi-stable critical point appears and then bifurcates into two critical points for smaller values of a.

The critical curve is color coded according to the type of critical point that it represents. The bifurcation diagram takes place in the a-y plane. It's the curve defined by the equation y prime equals 0. In this case, that's minus 1/4 plus a*y minus y squared.