

PROFESSOR: Hi everyone. Welcome back. So in this problem, I'd like to take a look at autonomous equations and phase lines. And specifically, we're going to take a look at the simple equation $\dot{x} = ax + 1$, which models births and death rates and a fixed replenishment rate for a population. So in this case, the variable a represents births minus deaths in a population. And 1 represents the constant input of new creatures.

So for part A, we're asked to find intervals of the variable a which determine the long-time stability of the population. And then secondly, for each typical case of a in the first part of the problem, we're asked to sketch a phase line and then also to sketch the solution x of t versus t . So I'll let you take a look at this problem, and I'll be back in a moment.

Hi Everyone. Welcome back. So the problem we're looking at is $\dot{x} = ax$. And as mentioned in the title of the problem, this is an autonomous equation which means that the right-hand side is not a function of time. The right-hand side only depends on x . And for these types of problems, the critical points or the points at which the right-hand side vanishes tend to determine the long-time characteristics of the solution.

So we can kind of get a handle on what intervals of a determine the long-time stability by sketching what \dot{x} versus x might look like. And we see that if a is positive, this represents a line with a positive slope a . So on the right-hand side of the intercept with the x -axis, \dot{x} is positive, which means that when the line goes through the point $\dot{x} = 0$, the term $ax + 1$ is going to be positive. I'll just point out that this point right here is when \dot{x} is equal to 0 . And this happens when x is equal to $-1/a$. So this point right here is $-1/a$. And when x is below $-1/a$, \dot{x} is negative, which means that solutions will want to move away from the point $-1/a$.

What other qualitative behavior could we have? Well, when a is below 0 , when a is less than 0 , the slope of this curve is going to be negative. So I'll plot again \dot{x} versus x . And again, the intercept is going to be at $-1/a$. However, when x is above the critical point, \dot{x} is negative. So solutions will want to tend to move back towards $-1/a$. And when x is below the critical point, we see that \dot{x} is in the upper plane, which means it's positive. So solutions will want to grow.

So we have two cases right now. One is when a is bigger than 0 . One is when a is less than 0 .

And then, of course, there's going to be an intermediate point, which is when a is equal to 0. And in this case, the curve x dot versus x , it's just going to be a straight line that goes through 1.

So this gives us three regions that will determine the long-time behavior. So the three regions are a equals 0, a less than 0, and then a bigger than 0. So this concludes part A. And for part B, we're asked to investigate the behavior of each of these three regions by constructing a phase line, and then, secondly, by sketching a solution or several solutions.

So for part B, let's take a look at the case when a is bigger than 0 first. And I'm just going to pick one value of a . I'll just pick a is equal to 1 to work things out concretely. And we're asked to sketch a phase line. So plot the phase line x here. And we know from the ODE that the critical point occurs at negative $1/a$. So in this case, there's going to be one critical point, and it's going to be at negative 1.

Now when x is bigger than negative 1, we've already seen that x dot is going to be positive. So when x is bigger than negative 1, in this region, x dot is positive. So above negative 1, x dot is positive, which means that if a solution starts at a point above negative 1, it's going to constantly increase forever. And likewise, if it's below negative 1, it's going to decrease forever. So this is x dot is negative in this region. And of course, x dot is 0 at the critical point, which means if we start at the critical point, we stay there for all time. So let's sketch some solutions.

So whenever I have to sketch solutions for a phase line, the first solution I always plot is going to be the critical point. So note that x equals negative 1 solves the differential equation. So x equals a critical point always solves the differential equation. So that means when x is equal to negative 1, we solve the differential equation. When x is bigger than negative 1, we attain exponential growth. And when x is less than negative 1, we have exponential growth in the opposite direction.

So this is sometimes said to be an unstable critical point, because any small perturbation away from negative 1 will make the solution diverge. Also note that once I have one of these curves, I can generate all the others by just picking it up and shifting it over. And the reason this works is because the original equation is autonomous, meaning that the right-hand side didn't depend on time. So whenever you have an autonomous equation, when you sketch one solution, you can always get the others by just picking-- or you can always get a family of

others by picking up that solution and just shifting it to the right and to the left.

Now when a is below 0, I'll just pick the value of a equals negative 1, we have a critical point at 1. And in this case, the arrows are going to point towards 1 like this. And again, I can sketch some curves. So I'll first draw the solution when x is equal to 1. And in this case, solutions are going to converge towards the critical point. And in this case, we say that the critical point x equals 1 is stable. And that's because any small perturbation in the solution will eventually just come back to the critical point at x equals 1.

And now lastly, we have the third case which is when a equals 0. And in this case, there are no critical points. In fact, \dot{x} was equal to 1. So \dot{x} is positive for all values of x . So our phase line is just a line with arrows. And in this case, if I were to sketch some solutions, I'll just draw my axes, x and t , there are no critical points. x is just increasing. So the solutions are going to look something like this. They're just going to be a family of straight lines.

So we've just sketched some phase lines, and we sketched several solutions for each phase line. We've seen how the critical points and the arrows around the critical points affect the long time behavior of the solutions. And this is typically the process that you go through when looking at autonomous equations. You always first, A, find the critical points, then B, find the regions between the critical points and figure out if \dot{x} is either positive or negative. And then that automatically gives you the phase line. Once you have the phase line, you can always just sketch a family of solutions. So I'd like to conclude here. And I'll see you next time.