Solutions that Blow Up: The Domain of a Solution

Example 1. Solve the IVP $\dot{y} = y^2,\ \ y(0) = 1$.

Solution. We can solve this using separation of variables.

Separate: $\dfrac{dy}{y^2} = dx$.

Integrate: $-1/y = x + C$.

Solve for y: $y = -1/(x + C)$.

Find C using the IC: $y(0) = 1 = -1/C$, therefore $C = -1$.

Solution: $y = -1/(x - 1) = 1/(1 - x)$.

The graph has a vertical asympote at $x = 1$.

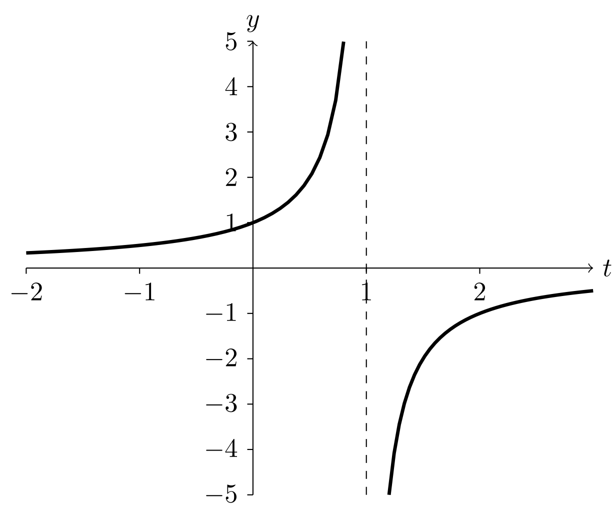

Fig. 1. Graph of $y = 1/(1 - x)$.

Starting at $x = 0$ the graph goes to infinity as $x \to 1$. Informally, we say *y blows up at $x = 1$*. The graph has two pieces. One is defined on $(-\infty, 1)$ and the other is defined on $(1, \infty)$. For technical reasons we prefer to say that we actually have *two* solutions to the DE. We indicate this by carefully specifying the domain of each.

$$y(x) = 1/(1 - x) \qquad x \text{ in the interval } (-\infty, 1) \tag{1}$$

$$y(x) = 1/(1 - x) \qquad x \text{ in the interval } (1, \infty). \tag{2}$$

Thus, the solution to the IVP in this example is solution (1).

The rule being followed here is that *solutions to ODE's have domain consisting of a single interval.* The example shows one reason for this: starting at $(0, 1)$ on solution (1) there is no way to follow the solution continuously to solution (2).