18.034 Honors Differential Equations
Spring 2009

LECTURE 10. MAXIMUM PRINCIPLE

By considering the points where a function attains a maximum or minimum one can get a good deal of information about the solutions of a differential equation without solving it.

We begin with a review of terminology. A point x is *interior* to an interval I if x is in I but is not an endpoint. An interval is *bounded* if its length is finite and *closed* if it contains its endpoints. For example, $1 < x \leqslant 2$ is bounded but not closed and $1 \leqslant x \leqslant 2$ is bounded and closed, $x > 1$ is neither bounded nor closed.

A real-valued function $f(x)$ defined on the interval I of x is said to have a *maximum* at x_0 if $x_0 \in I$ and $f(x_0) \geqslant f(x)$ on $x \in I$. The maximum is called interior if x_0 is an interior point of I and it is called positive if $f(x_0) > 0$. The "negative interior minimum" is defined similarly.

If the inequality $f(x_0) \geqslant f(x)$ is required to hold only in some open interval $J \subset I$, with $x \in J$ and $x_0 \in J$ then the maximum is said to be *local*. In what follows, we use the terms *maximum* and *minimum* to mean local maximum and local minimum, respectively.

We recall the following theorem of calculus.

Theorem 10.1. *A continuous real-valued function on a bounded closed interval attains its maximum and minimum on the interval.*

At an interior maximum or minimum x_0 a differentiable function f satisfies $f'(x_0) = 0$. If f is twice differentiable then f satisfies the additional condition $f''(x_0) \leqslant 0$ at a maximum and $f''(x_0) \geqslant 0$ at a minimum. Thus, at a positive interior maximum a twice-differentiable function f satisfies

$$f(x_0) > 0, \qquad f'(x_0) = 0, \quad \text{and} \quad f''(x_0) \leqslant 0. \tag{10.1}$$

At a negative interior minimum it satisfies

$$f(x_0) < 0, \qquad f'(x_0) = 0, \quad \text{and} \quad f''(x_0) \geqslant 0. \tag{10.2}$$

Our aim here is to show how (10.1) and (10.2) are used to obtain information about the solutions of differential equations. We adopt the convention that $a < b$ and that y is continuous on the closed interval $a \leqslant x \leqslant b$ or $a \leqslant x < \infty$, as the case may be, which is implicitly specified in the statement of the problem. The function y is also twice differentiable on the interior.

Example 10.2. Show that the only solution of the boundary value problem

$$y'' + e^x y' = (x^2 + 1)y, \qquad y(a) = y(b) = 0$$

is $y = 0$.

SOLUTION. If y is not identically zero there must be a positive maximum or a negative minimum at some point c where $a < c < b$. At this point $y'(c) = 0$ and the differential equation becomes

$$y''(c) = (c^2 + 1)y(c).$$

So, either condition (10.1) or (10.2) leads to a contradiction.

Equations (10.1) and (10.2) constitute the *maximum principle* in its broadest sense. In a narrower sense, the term "maximum principle" is used to describe an estimate for y on $a < x < b$ from knowledge of the boundary values $y(a)$ and $y(b)$.

Example 10.3. Suppose that y solves the differential equation

$$(\cos x)^2 y'' + x^2(1-x)y' = 2y$$

on the interval $a \leqslant x \leqslant b$ and $|y(a)|, |y(b)| \leqslant m$, where m is a positive constant. Show that $|y(x)| < m$ for $a < x < b$.

SOLUTION. If otherwise, either y has a positive interior maximum or a negative interior minimum. Either condition leads to a contradiction as in Example 10.2.

It is a consequence of Rolle's theorem that if $y' > 0$ on an interval then y is strictly increasing. The relation $y' > 0$ is an example of a *differential inequality*. Differential inequalities form a major subfield of the modern theory of differential equations.

Example 10.4. Show that a function y satisfying

$$e^x y'' + y' \sin x - (1+x)y \geqslant 0, \qquad \text{for} \quad x > 0$$

and $y(0) \geqslant 0$, $y'(0) > 0$ must be strictly increasing.

SOLUTION. If otherwise, there are points x_1 and x_2 such that $0 < x_1 < x_2$ and $y(x_1) \geqslant y(x_2)$. It is obvious from a sketch that y attains a positive maximum at an interior point c of the interval $0 \leqslant x \leqslant x_2$. (An analytic proof can be given by use of Theorem 10.1.) At the maximum we have $y'(c) = 0$, and the differential inequality becomes

$$e^c y''(c) - (1+c)y(c) \geqslant 0.$$

Then, (10.1) leads to a contradiction.

A *zero* of a function y is a point where $y(x) = 0$. In general, a solution of a second-order linear differential equation cannot have zeros too close together unless it is identically zero.

Example 10.5. Let y satisfy

$$y'' + e^{-x}y' = y \sin x, \qquad x > 0$$

and $y(0) = y(b) = 0$ where $b > 0$. Show that $b > \pi$ unless y is identically zero.

SOLUTION. Suppose that $b \leqslant \pi$ and that y is not identically zero for $0 < x < b$. Then there is a positive maximum or negative minimum at an interior point c of this interval. At c the differential equation becomes $y''(c) = y(c) \sin c$ and either condition (10.1) or (10.2) lead to a contradiction. Hence $y = 0$ on the interval $0 < x < b$. This gives $y(b) = y'(b) = 0$ and then $y = 0$ by the uniqueness theorem for second-order initial value problems.

Exercises.

1. Show that no nontrivial solution of

$$w'' + \lambda q(x) w = 0, \qquad w(a) = 0 = w(b)$$

exists if $q(x) > 0$ and $\lambda < 0$.

2. Consider

$$e^{\cos x} w'' - x^2 w + x^3 = 0, \qquad \text{for all } x,$$

where $w(0) = 0$. Show that $y = w - x$ have a positive maximum or negative minimum at any value $x = c$. Show also that the sign of w'' is the same as the sign of $w - x$.

3. Show that the solution of the DE $(\cosh x)y'' + (\cos x)y' = (1+x^2)y$ for $a < x < b$ with $y(a) = 1 = y(b)$ satisfies $0 < y(x) < 1$ for $a < x < b$.