18.034 Honors Differential Equations
Spring 2009

18.034 Practice Final

Notation. $' = d/dt$.

1. (a) Is the differential form $x^2y^3dx + x(1+y^2)dy$ exact?

(b) Find a function $\mu(x,y)$ so that $\mu(x,y)(x^2y^3dx + x(1+y^2)dy)$ becomes exact.

(c) Solve the differential equation

$$\frac{dy}{dx} = -\frac{x^2y^3}{x(1+y^2)}.$$

2. (a) A basis of solutions of the differential equation

$$y'' + \frac{2t}{t^2-1}y' - 16\frac{1}{(t^2-1)^2}y = 0$$

is given by

$$y_1(t) = \left(\frac{t-1}{t+1}\right)^2, \qquad y_2(t) = \left(\frac{t+1}{t-1}\right)^2.$$

Compute the Wronskian $W(y_1, y_2)$.

(b) Use variation of parameters to find a particular solution of

$$y'' + \frac{2t}{t^2-1}y' - 16\frac{1}{(t^2-1)^2}y = t^2 - 1.$$

3. Show that the initial value problem $y' = |y|^{1/2}$ and $y(0) = -1$ is well-posed on $t \in [0, a)$ if $a \leq 2$ but not if $a > 2$.

4. Solve the initial value problem

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}, \qquad \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

(a) by using the eigenvalues and (b) by the Laplace transform.

5. Consider the plane autonomous system

$$x' = y, \qquad y' = x^2 - y - \epsilon,$$

where ϵ is a real parameter.

(a) Find the critical points.

(b) If the system has critical points, then discuss the behavior of the solutions near the critical points.

(c) Discuss how the behavior of solutions changes with ϵ.

6. (a) The Liénard euqation is

$$u'' + c(u)u' + g(u) = 0,$$

where $c(u) \geq 0$ and $g(0) = 0$, $ug(u) > 0$ for $u \neq 0$ and small. Show that the critical point $u = 0$ and $u' = 0$ is stable.

(b) When $c(u) = 1$, show that the critical point $u = 0$ and $u' = 0$ is asymptotically stable.

7. (a) Show that every solution of the plane autonomous system

$$x' = ye^{1+x^2+y^2}, \qquad y' = -xe^{1+x^2+y^2}$$

is periodic.

(b) Show that the system

$$x' = x - x^3 - xy^2, \qquad y' = y - y^3 - yx^2$$

has a unique limit cycle.

(c) Show that the system

$$x' = x - xy^2 + y^3, \qquad y' = 3y - yx^2 + x^3$$

has no nontrivial periodic solution lying inside the circle $x^2 + y^2 = 4$.