18.03 Recitation 19, April 15, 2010

Laplace transform II

1. Find the unit impulse response for $D + 3I$.

$x' + 3x = \delta$. Taking the Laplace transform of both sides leads to $s\mathfrak{X}(s) + 3\mathfrak{X}(s) = 1$, and $\mathfrak{X}(s) = \frac{1}{s+3}$. The inverse Laplace transform of $\frac{1}{s+3}$ is $x(t) = e^{-3t}$.

2. Find the solution to $\dot{x} + 3x = e^{-t}$ with rest initial condition (so $x(0) = 0$) using Laplace transform.

The solution satisfies the rest initial condition, so we can take the Laplace transform of both sides directly and get $s\mathfrak{X}(s) + 3\mathfrak{X}(s) = \frac{1}{s+1}$. So $\mathfrak{X}(s) = \frac{1}{(s+1)(s+3)}$. Assume $\frac{1}{(s+1)(s+3)} = \frac{a}{s+1} + \frac{b}{s+3}$, then $1 = a(s+3) + b(s+1)$. Set $s = -1$, then $a = 1/2$; set $s = -3$, then $b = -1/2$. So $\frac{1}{(s+1)(s+3)} = \frac{1}{2} \cdot \frac{1}{s+1} - \frac{1}{2} \cdot \frac{1}{s+3}$, and its inverse Laplace transform is $\frac{1}{2}(e^{-t} - e^{-3t})$.

3. Find the unit impulse response for $D^3 + D$ using Laplace transform.

$x^{(3)} + x' = \delta$. Taking the Laplace transform, we get $s^3\mathfrak{X}(s) + s\mathfrak{X}(s) = 1$, and hence $\mathfrak{X}(s) = \frac{1}{s^3+s} = \frac{1}{s(s^2+1)}$. Notice that $\frac{1}{s(s^2+1)} = \frac{1}{s} - \frac{s}{s^2+1}$. So $x(t) = 1 - \cos t$.

4. (a) Find the solution to $\dot{x}_r + 3x = 1$ such that $x(0+) = 2$ using the formula for $\mathcal{L}[f_r'(t)]$.

By $t-$derivative rule for $\dot{x}_r$, the Laplace transform of the LHS of the equation is $s\mathfrak{X}(s) - 2 + 3\mathfrak{X}(s)$, and $\mathcal{L}[1] = \frac{1}{s}$. So $\mathfrak{X}(s) = \frac{(1/s)+2}{s+3} = \frac{1}{s(s+3)} + \frac{2}{s+3}$. Moreover, $\frac{1}{s(s+3)} = \frac{1}{3}(\frac{1}{s} - \frac{1}{s+3})$, so $x(t) = \frac{1}{3}(1 - e^{-3t}) + 2e^{-3t} = \frac{1}{3} + \frac{5}{3}e^{-3t}$.

(b) The jump in value of $x(t)$ at $t = 0$ can be created by adding $2\delta(t)$ to the right hand side of the equation. Using the expression for $\mathcal{L}[f'(t)]$, find the solution to $\dot{x} + 3x = 1 + 2\delta(t)$ such that $x(0-) = 0$.

By $t-$derivative rule for x', the the Laplace transform of the LHS of the equation is $s\mathfrak{X}(s) + 3\mathfrak{X}(s)$, and the RHS is $\frac{1}{s} + 2$. So $\mathfrak{X}(s) = \frac{(1/s)+2}{s+3}$, which is the same with (a). So the solution is $x(t) = u(t)(\frac{1}{3} + \frac{5}{3}e^{-3t})$.