

Recitation 8, March 2, 2010

Homogeneous 2nd order linear constant coefficient equations.

Solution suggestions

1. Start with $\ddot{x} + \omega^2 x = 0$. What is the characteristic polynomial? What are its roots? What are the exponential solutions? What are the real and imaginary parts of this solution?

The characteristic polynomial is $p(s) = s^2 + \omega^2$. Its roots are $\pm i\omega$. The exponential solutions are $k_1 e^{i\omega t}$ and $k_2 e^{-i\omega t}$. The real part is $C_1 \cos(\omega t)$, and the imaginary part is $C_2 \sin(\omega t)$.

2. Suppose that $e^{-t/2} \cos(3t)$ is a solution of the equation $m\ddot{x} + b\dot{x} + kx = 0$.

(a) What can you say about m, b, k ?

We can write $e^{-t/2} \cos(3t) = \operatorname{Re} e^{(-1/2 \pm 3i)t}$, so $p(s) = ms^2 + bs + k$ has solutions $-\frac{1}{2} \pm 3i$. This means $p(s) = m(s + 1/2 - 3i)(s + 1/2 + 3i) = m(s^2 + s + \frac{37}{4})$. Then m is any number, $b = m$, and $k = \frac{37}{4}m$.

(b) What is an exponential solution of this differential equation?

We can just take an exponential function whose real part is $e^{-t/2} \cos(3t)$. As seen in (a), one possible exponential solution is $e^{(-1/2 + 3i)t}$.

(c) Sketch the curve in the complex plane traced by one of the exponential solutions. Then sketch the graph of the real part, and explain the relationship.

The curve is a spiral. As t increases, it moves counterclockwise around the origin, and approaches zero. The graph of the real part describes a damped oscillation, i.e., it is roughly sinusoidal, with exponentially decreasing amplitude. Introduce the time axis to the complex plane, and stretch out the trajectory according to time. Projecting this trajectory onto the (real axis)-(time axis) plane yields the graph of the real part.

(d) What is the general solution?

The general solutions are given by real linear combinations of the real part and the imaginary part of $e^{(-1/2 + 3i)t}$, i.e., $C_1 e^{-t/2} \cos(3t) + C_2 e^{-t/2} \sin(3t)$ for any real numbers C_1 and C_2 , or equivalently, $e^{-t/2} A \cos(3t - \phi)$, where $A \geq 0$ and $\phi \in [0, 2\pi]$.

3. A damped sinusoid $x(t) = Ae^{-at} \cos(\omega t)$ has “pseudo-period” $2\pi/\omega$. The pseudo-period, and hence ω , can be measured from the graph: it is twice the distance between successive zeros of $x(t)$. What is the spacing between successive maxima of $x(t)$? Is it always the same, or does it differ from one successive pair of maxima to the next?

The extrema of $x(t) = Ae^{-at} \cos(\omega t)$ occur when $\dot{x}(t) = 0$, i.e., $-a \cos(\omega t) = \omega \sin(\omega t)$. Let's assume $\omega \neq 0$, otherwise $x(t)$ has at most one maximum. When $\omega \neq 0$, the extrema are achieved at t where $\tan(\omega t) = -a/\omega$. Since minima and maxima of $x(t)$ are alternating, the maxima occur at every other t such that $\tan(\omega t) = -a/\omega$. If t_0 and t_1 are

successive maxima, then $t_1 - t_0 =$ twice the period of $\tan(\omega t) = 2\pi/\omega$, and it's always the same since it doesn't depend on t or x .

4. Suppose that successive maxima of $x(t) = Ae^{-at} \cos(\omega t)$ occur at $t = t_0$ and $t = t_1$. What is the ratio $x(t_1)/x(t_0)$? (Hint: Compare $\cos(\omega t_0)$ and $\cos(\omega t_1)$.) Does this offer a means of determining the value of a from the graph?

As seen in question 3, $t_1 - t_0 =$ twice the period of $\tan(\omega t) = 2\pi/\omega$. So $\cos(\omega t_1) = \cos(\omega t_0 + 2\pi) = \cos(\omega t_0)$. The ratio is given by $x(t_1)/x(t_0) = e^{-a(t_1-t_0)}$. Therefore, $a = \frac{\ln x(t_0) - \ln x(t_1)}{t_1 - t_0}$.

5. For what value of b does $\ddot{x} + b\dot{x} + x = 0$ exhibit critical damping? For this value of b , what is the solution x_1 with $x_1(0) = 1$, $\dot{x}_1(0) = 0$? What is the solution x_2 with $x_2(0) = 0$, $\dot{x}_2(0) = 1$? (This is a "normalized pair" of solutions.) What is the solution such that $x(0) = 2$ and $\dot{x}(0) = 3$?

The characteristic polynomial is $p(s) = s^2 + bs + 1$. To exhibit critical damping, this must be a square, i.e., $(s - k)^2$ for some k . Multiplying and comparing yields $-2k = b$ and $k^2 = 1$. Therefore, $b = \pm 2$. When $b = -2$, e^t is a solution, and it exhibits exponential growth instead of damping, so we reject that value of b . When $b = 2$, the general solutions are given by $x(t) = (C_0 + C_1 t)e^{-t}$, and $x(0) = C_0$, $\dot{x}(0) = -C_0 + C_1$. For x_1 , we set $C_0 = 1$ and $C_1 = 1$; for x_2 , we set $C_0 = 0$ and $C_1 = 1$; for x_3 , $C_0 = 2$ and $C_1 = 5$.

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