12. Resonance and the exponential shift law

12.1. **Exponential shift.** The calculation (10.1)

$$p(D)e^{rt} = p(r)e^{rt} \tag{1}$$

extends to a formula for the effect of the operator $p(D)$ on a product of the form $e^{rt}u$, where u is a general function. This is useful in solving $p(D)x = f(t)$ when the input signal is of the form $f(t) = e^{rt}q(t)$.

The formula arises from the product rule for differentiation, which can be written in terms of operators as

$$D(vu) = v\,Du + (Dv)u.$$

If we take $v = e^{rt}$ this becomes

$$D(e^{rt}u) = e^{rt}Du + re^{rt}u = e^{rt}(Du + ru)\,.$$

Using the notation I for the identity operator, we can write this as

$$D(e^{rt}u) = e^{rt}(D + rI)u. \tag{2}$$

If we apply D to this equation again,

$$D^2(e^{rt}u) = D(e^{rt}(D + rI)u) = e^{rt}(D + rI)^2u\,,$$

where in the second step we have applied (2) with u replaced by $(D + rI)u$. This generalizes to

$$D^k(e^{rt}u) = e^{rt}(D + rI)^k u.$$

The final step is to take a linear combination of D^k's, to form a general LTI operator $p(D)$. The result is the

Exponential Shift Law:

$$\boxed{p(D)(e^{rt}u) = e^{rt}p(D + rI)u} \tag{3}$$

The effect is that we have pulled the exponential outside the differential operator, at the expense of changing the operator in a specified way.

12.2. **Product signals.** We can exploit this effect to solve equations of the form

$$p(D)x = e^{rt}q(t)\,,$$

by a version of the method of variation of parameter: write $x = e^{rt}u$, apply $p(D)$, use (3) to pull the exponential out to the left of the operator, and then cancel the exponential from both sides. The result is

$$p(D + rI)u = q(t)\,,$$

a new LTI ODE for the function u, one from which the exponential factor has been eliminated.

Example 12.2.1. Find a particular solution to $\ddot{x} + \dot{x} + x = t^2 e^{3t}$.

With $p(s) = s^2 + s + 1$ and $x = e^{3t}u$, we have

$$\ddot{x} + \dot{x} + x = p(D)x = p(D)(e^{3t}u) = e^{3t}p(D + 3I)u\,.$$

Set this equal to $t^2 e^{3t}$ and cancel the exponential, to find

$$p(D + 3I)u = t^2$$

or $\dot{u} + 3u = t^2$. This is a good target for the method of undetermined coefficients (Section 11). The first step is to compute

$$p(s + 3) = (s + 3)^2 + (s + 3) + 1 = s^2 + 7s + 13\,,$$

so we have $\ddot{u} + 7\dot{u} + 13u = t^2$. There is a solution of the form $u_p = at^2 + bt + c$, and we find it is

$$u_p = (1/13)t^2 - (14/13^2)t + (85/13^3)\,.$$

Thus a particular solution for the original problem is

$$x_p = e^{3t}((1/13)t^2 - (14/13^2)t + (85/13^3))\,.$$

Example 12.2.2. Find a particular solution to $\dot{x} + x = te^{-t}\sin t$.

The signal is the imaginary part of $te^{(-1+i)t}$, so, following the method of Section 10, we consider the ODE

$$\dot{z} + z = te^{(-1+i)t}\,.$$

If we can find a solution z_p for this, then $x_p = \operatorname{Im} z_p$ will be a solution to the original problem.

We will look for z of the form $e^{(-1+i)t}u$. The Exponential Shift Law (3) with $p(s) = s + 1$ gives

$$\begin{aligned}\dot{z} + z &= (D + I)(e^{(-1+i)t}u) = e^{(-1+i)t}((D + (-1 + i)I) + I)u \\ &= e^{(-1+i)t}(D + iI)u.\end{aligned}$$

When we set this equal to the right hand side we can cancel the exponential:

$$(D + iI)u = t$$

or $\dot{u} + iu = t$. While this is now an ODE with *complex* coefficients, it's easy to solve by the method of undetermined coefficients: there is a solution of the form $u_p = at + b$. Computing the coefficients, $u_p = -it + 1$; so $z_p = e^{(-1+i)t}(-it + 1)$.

Finally, extract the imaginary part to obtain x_p:

$$z_p = e^{-t}(\cos t + i\sin t)(-it + 1)$$

has imaginary part

$$x_p = e^{-t}(-t\cos t + \sin t).$$

12.3. **Resonance.** We have noted that the Exponential Response Formula for a solution to $p(D)x = e^{rt}$ fails when $p(r) = 0$. For example, For example, suppose we have $\dot{x} + x = e^{-t}$. The Exponential Response Formula proposes a solution $x_p = e^{-t}/p(-1)$, but $p(-1) = 0$ so this fails. There is no solution of the form ce^{rt}.

This situation is called *resonance*, because the signal is tuned to a natural mode of the system.

Here is a way to solve $p(D)x = e^{rt}$ when this happens. The ERF came from the calculation

$$p(D)e^{rt} = p(r)e^{rt},$$

which is valid whether or not $p(r) = 0$. We will take this expression and *differentiate it with respect to* r, keeping t constant. The result, using the product rule and the fact that partial derivatives commute, is

$$p(D)te^{rt} = p'(r)e^{rt} + p(r)te^{rt}$$

If $p(r) = 0$ this simplifies to

$$p(D)te^{rt} = p'(r)e^{rt}\,. \tag{4}$$

Now if $p'(r) \neq 0$ we can divide through by it and see:

The Resonant Exponential Response Formula: If $p(r) = 0$ then a solution to $p(D)x = ae^{rt}$ is given by

$$\boxed{x_p = a\,\frac{te^{rt}}{p'(r)}} \tag{5}$$

provided that $p'(r) \neq 0$.

In our example above, $p(s) = s + 1$ and $r = 1$, so $p'(r) = 1$ and $x_p = te^{-t}$ is a solution.

This example exhibits a characteristic feature of resonance: the solutions grow faster than you might expect. The characteristic polynomial leads you to expect a solution of the order of e^{-t}. In fact the solution is t times this. It still decays to zero as t grows, but not as fast as e^{-t} does.

Example 12.3.1. Suppose we have a harmonic oscillator represented by $\ddot{x} + \omega_n^2 x$, or by the operator $D^2 + \omega_n^2 I = p(D)$, and drive it by the signal $a\cos(\omega t)$. This ODE is the real part of

$$\ddot{z} + \omega_n^2 z = ae^{i\omega t}\,,$$

so the Exponential Response Formula gives us the periodic solution

$$z_p = a\frac{e^{i\omega_n t}}{p(i\omega)}\,.$$

This is fine *unless* $\omega = \omega_n$, in which case $p(i\omega_n) = (i\omega_n)^2 + \omega_n^2 = 0$; so the amplitude of the proposed sinusoidal response should be infinite. The fact is that there is *no* periodic system response; the system is in *resonance* with the signal.

To circumvent this problem, let's apply the Resonance Exponential Response Formula: since $p(s) = s^2 + \omega_n^2$, $p'(s) = 2s$ and $p'(i\omega_n) = 2i\omega_0$, so

$$z_p = a\frac{te^{i\omega_n t}}{2i\omega_n}\,.$$

The real part is

$$x_p = \frac{a}{2\omega_n} t\sin(\omega_n t)\,.$$

The general solution is thus

$$x = \frac{a}{2\omega_n} t\sin(\omega_n t) + b\cos(\omega_n t - \phi)\,.$$

In words, all solutions oscillate with pseudoperiod $2\pi/\omega_n$, and grow in amplitude like $at/(2\omega_n)$. When ω_n is large—high frequency—this rate of growth is small.

12.4. **Higher order resonance.** It may happen that both $p(r) = 0$ and $p'(r) = 0$. The general picture is this: Suppose that k is such that $p^{(j)}(r) = 0$ for $j < k$ and $p^{(k)}(r) \neq 0$. Then $p(D)x = ae^{rt}$ has as solution

$$x_p = a\frac{t^k e^{rt}}{p^{(k)}(r)}\,. \tag{6}$$

For instance, if $\omega = \omega_0 = 0$ in Example 12.3.1, $p'(i\omega) = 0$. The signal is now just the constant function a, and the ODE is $\ddot{x} = a$. Integrating twice gives $x_p = at^2/2$ as a solution, which is a special case of (6), since $e^{rt} = 1$ and $p''(s) = 2$.

You can see (6) in the same way we saw the Resonant Exponential Response Formula. So take (4) and differentiate again with respect to r:

$$p(D)t^2e^{rt} = p''(r)e^{rt} + p'(r)te^{rt}$$

If $p'(r) = 0$, the second term drops out and if we suppose $p''(r) \neq 0$ and divide through by it we get

$$p(D)\left(\frac{t^2 e^{rt}}{p'(r)}\right) = e^{rt}$$

which the case $k = 2$ of (6). Continuing, we get to higher values of k as well.

12.5. **Summary.** The work of this section and the last can be summarized as follows: Among the responses by an LTI system to a signal which is polynomial times exponential (or a linear combination of such) there is always one which is again a linear combination of functions which are polynomial times exponential. By the magic of the complex exponential, sinusoidal factors are included in this.