1. The Tacoma Narrows Bridge: resonance vs flutter

On July 1, 1940, a bridge spanning the Tacoma Narrows opened to great celebration. It dramatically shortened the trip from Seattle to the Kitsap Peninsula. It was an elegant suspension bridge, a mile long (third longest in the US at the time) but just 39 feet across. Through the summer and early fall, drivers noticed that it tended to oscillate vertically, quite dramatically. It came to be known as "Galloping Gertie." "Motorists crossing the bridge sometimes experienced "roller-coaster like" travel as they watched cars ahead almost disappear vertically from sight, then reappear."[1]

During the first fall storm, on November 7, 1940, with steady winds above 40 mph, the bridge began to exhibit a different behavior. It *twisted*, part of one edge rising while the opposing edge fell, and then the reverse. At 10:00 AM the bridge was closed. The torsional oscillations continued to grow in amplitude, till, at just after 11:00, the central span of the bridge collapsed and fell into the water below. One car and a dog were lost.

Why did this collapse occur? Were the earlier oscillations a warning sign? Many differential equations textbooks announce that this is an example of *resonance*: the gusts of wind just happened to match the natural frequency of the bridge.

The problem with this explanation is that the wind was not gusting—certainly not at anything like the natural frequency of the bridge. This explanation is worthless.

Structural engineers have studied this question in great detail. They had determined already before the bridge collapsed that the vertical oscillation was self-limiting, and not likely to lead to a problem. The torsion oscillation was different. To model it, pick a portion of the bridge far from the support towers. Let $\theta(t)$ denote its angle off of horizontal, as a function of time. The torsional dynamics can be modeled by a second order differential equation of the form

$$\ddot{\theta} + b\dot{\theta} + \omega_n^2\theta = F$$

where ω_n^2 is the natural circular frequency of the torsional oscillation, and b is a damping term. The forcing term F depends upon θ itself, and its derivatives. To a reasonable approximation we can write

$$F = a_0\theta + a_1\dot{\theta}$$

where a_0 and a_0 are functions of the wind velocity v which are determined by the bridge characteristics. The resulting differential equation

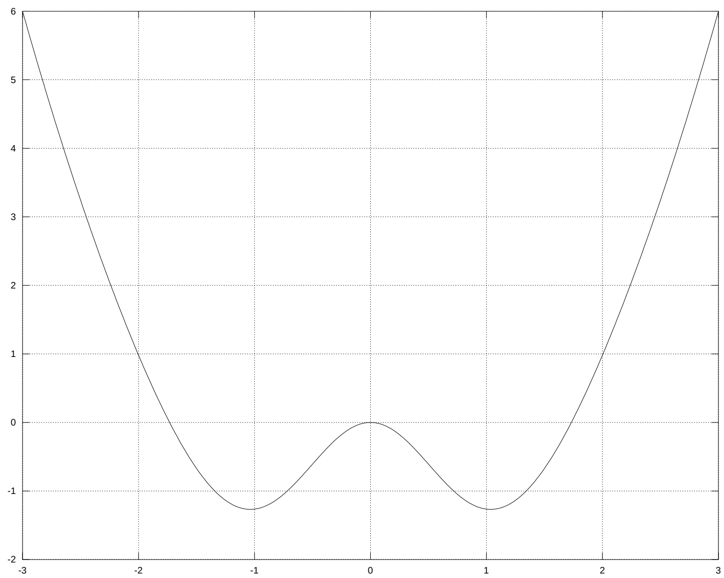

FIGURE 1. Dependence of the forcing term on wind velocity

is analogous to the equation governing the behavior of the mass in a spring/mass/dashpot system which is driven through both the mass and the dashpot—except that that "input signal," which is the position of the forcing plate in the spring/mass/dashpot system, is now the output signal, the angular deflection itself. This is an instance of "self-excitation."

Notice that this equation can be rewritten as

$$\ddot{\theta} + (b - a_1)\dot{\theta} + (\omega_n^2 - a_0)\theta = 0 \tag{1}$$

It turns out that in the case of the Tacoma Narrows bridge the value of a_0 is small relative to ω_n^2; the effect is to slightly alter the effective natural frequency of torsional oscillation. For simplicity we'll just suppose it's negligable and drop it.

The function $a_1(v)$ reflects mainly turbulence effects. The technical term for this effect is *flutter*. The same mechanism makes flags flap and snap in the wind. It turns out that the graph of $a_1(v)$ has the following shape.

When $|v|$ is small, $a_1(v) < 0$: the wind actually increases the damping of the bridge; it becomes *more* stable. When $|v|$ is somewhat larger,

$a_1(v) = 0$, and the wind has no damping effect. When $|v|$ increases still more, it starts to erode the damping of the bridge, till, when v hits a certain critical value, it overwhelms the intrinsic damping of the bridge. The result is *anti-damping*, a negative effective damping constant. For the Tacomah Narrows Bridge, the critical value of velocity was discovered, on November 7, 1940, to be around 40 miles per hour.

Solutions to (1) are linear combinations of the functions e^{rt} where r is a root of the characteristic polynomial $p(s) = s^2 + (b - a_1)s + \omega_n^2$:

$$r = -\frac{b - a_1}{2} \pm \sqrt{\frac{(b - a_1)^2}{4} - \omega_n^2}$$

The movies of the bridge collapse clearly show large oscillations, so in this regime $|b - a_1| < 2\omega_n$, square root is negative, and the roots have nonzero imaginary parts. The real part of each root is $k = (a_1 - b)/2$, and when v is such that $a_1(v) > b$ this is positive. If we write $r = k \pm i\omega$, the general solution is

$$\theta = Ae^{kt} \cos(\omega t - \phi)$$

Its peaks grow in magnitude, exponentially.

This spells disaster. There are compensating influences which slow down the rate of growth of the maxima, but in the end the system will—and did—break down.

References

[1] K. Y. Billah and R. H. Scanlan, Resonance, Tacomah Narrows bridge failure, and undergraduate physics textbooks, Am. J. Phys. 59 (1991) 118–124.