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The matrix exponential: initial value problems.

1. Definition of e^{At}
2. Computation of e^{At}
3. Uncoupled example
4. Defective example
5. Exponential law

[1] Recall from day one:

(a) x' = rx with initial condition x(0) = 1 has solution x = e^{rt} .

x' = rx with any initial condition has solution x = e^{rt} x(0) .

Later, we decided to *define* e^{it} as the solution to

(b) x' = ix with initial condition   x(0) = 1 .

Following Euler, a solution is given by   cos t + i sin t , so we found that

             e^{it} = cos(t) + i sin(t)

(c) Now we are studying u' = A u . Let's try to *define*

   The solution to u' = Au with initial condition   u(0)
   is   u = e^{At}u(0). (**)

Note that the initial value u(0) is a vector, and u(t) is a vector
valued function. So the expression e^{At} must denote a matrix, or
rather a matrix valued function.

What could e^{At} be? For a start, what is its first column?
Recall that the first column of any matrix B is the product B[1;0] ,
and [b_1   b_2] [1;0] = b_1 , so combining this with (**) we see:

The first column of e^{At} is the solution to u' = Au with
u(0) = [1;0].

Similarly,

The second column of e^{At} is the solution to u' = Au with
u(0)= [0;1].

This is the DEFINITION of e^{At} . It makes (**) true for all u(0) ,
because e^{At} u(0) is a solution (being a linear combination of the
columns of e^{At} , which are solutions), and when t = 0 we get

         e^{A0} u(0) = I u(0) = u(0) .

[2] Computation of e^{At}

We need a method for computing it, though. To explore this we'll use the

Example: A = [1 1 ; 0 2] .

This is upper triangular, so its eigenvalues are given by the diagonal entries: lambda = 1, lambda = 2. The (tr,det) pair lies in the upper right quadrant, below the critical parabola; the phase portrait is an unstable node.

Find eigenvectors:

  lambda_1 = 1 :   A - I : [0 1 ; 0 1] [?;?] = [0;0] : v_1 = [1;0]

  lambda_2 = 2 : A - 2I : [-1 1 ; 0 0] [?;?] = [0;0] : v_2 = [1;1]

Two independent solutions are given by

      u_1 = [e^t ; 0] , u_2 = [e^{2t} ; e^{2t}]

and the general solution is

      u = c_1 [e^t ; 0] + c_2 [e^{2t} ; e^{2t}]

We could go ahead and solve for c_1 and c_2 to get solutions with the desired initial conditions. What follows is a clever way to do that.

There is a compact way to write this linear combination: it is

      u = [e^t , e^{2t} ; 0 , e^{2t}] [c_1 ; c_2].   (***)

This matrix is a "fundamental matrix" for the system: its columns are independent solutions. Such a matrix will be denoted by Phi(t); so here

      Phi(t) = [e^t , e^{2t} ; 0 , e^{2t}]

Phi(t) behaves very much like we want e^{At} to behave; its columns are solutions, even independent ones, and the general solution is given by

      Phi(t) [c_1 ; c_2]

The matrix exponential e^{At} is a fundamental matrix: it is the fundamental matrix Phi(t) such that Phi(0) = I .

Our Phi(t) does not evaluate this way. To fix this, I claim we should form

      Phi(t) Phi(t)^{-1}

Explanation: If B is a square matrix, you can ask whether it has an *inverse* matrix, a matrix B^{-1} such that

       B B{-1} = I   and   B^{-1} B = I

(either implies both). The answer, as for numbers, is not always. It turns out that there is an inverse exactly when det(B) is not zero.

In the 2x2 case, B = [a b ; c d] and

      [a b ; c d]^{-1} = (1/det(A)) [d -b ; -c a].

We can check this:

      [a b ; c d] [d -b ; -c a] = [ad-bc 0 ; 0 ad-bc] = (det B) I

Now let's see: each column Phi(t) Phi(0)^{-1} is a linear combination
of the columns of Phi(t) , so it's a solution. What remains is to check
the normalization; but Phi(0) Phi(0)^{-1} = I .

Conclusion:

      e^{At} = Phi(t) Phi(0)^{-1}

where Phi(t) is ANY fundamental matrix for A.2A

In our example, Phi(0) = [1 1 ; 0 1] , Phi(0)^{-1} = [1 -1 ; 0 1] ,
and so

      e^{At} = [e^t e^{2t} ; 0 e^{2t}] [1 -1 ; 0 1]

              = [e^t , e^{2t} - e^t ; 0 , e^{2t}].

[3] Uncoupled example: Suppose A = [a 0 ; 0 d] .
The eigenvalues are lambda_1 = a and lambda_2 = d

I can see the eigenvectors: v_1 = [1 ; 0] , v_2 = [0 ; 1] .

Basic solutions are e^{lambda_1 t} [1 ; 0]
                     e^{lambda_2 t} [0 ; 1]

so Phi(t) = [e^{lambda_1 t} 0 ; 0 e^{lambda_2 t}]

and this is already normalized: so it is the matrix exponential.

[4] Defective example.

Sometimes the matrix exponential can be a bit unexpected. For example:

     A = [0 1 ; 0 0]

Then tr A = 0 and det A = 0 , so the only eigenvalue is 0 , with
multiplicity 2 . This is not a diagonal matrix, so it is defective,
and we could find solutions by the standard method. However, it is also
a companion matrix, for the second order equation   x" = 0 .
Solutions of this are easy! x_1 = 1 , x_2 = t . So basic solutions to

     u' = A u

are u_1 = [x_1 ; x_1'] = [1 ; 0]
     u_2 = [x_2 ; x_2'] = [t ; 1]

     Phi(t) = [1 t ; 0 1]

This satisfies Phi(0) = I , so

e^{At} = [1 t ; 0 1]

[5] We can take t to be a specific value, of course: eg t = 1 :

e^[0 1 ; 0 0] = [1 1 ; 0 1]

and this lets us define e^A for any square matrix A .

Then e^0 = I , as you might expect, but watch out:

e^A e^B = e^{A+B}   *provided that* AB = BA

So for example (e^A)^n = e^{nA}