18.03 Class 24, April 5, 2010

Unit impulse and step responses

1. Generalized derivative
2. Rest initial conditions
3. First order unit step/impulse response
4. Second order unit step/impulse response

[1] Generalized derivative

The unit step and delta functions help deal with events happening on a time scale which is very short relative to our interest.

u(t) can be thought of as a function which, except for t very near zero, is given by u(t) = 0 for t < 0 and u(t) = 1 for t > 0 .

delta(t) can be thought of as a function which is zero except for t very near zero, and has area under its graph equal to 1 .

u'(t) = delta(t) .

A generalized function is by definition a sum g(t) = g_r(t) + g_s(t) , where its *regular part* g_r(t) is piecewise smooth, and its *singular part* g_s(t) is a linear combination of shifted delta functions.

Any regular f(t) function has a "generalized derivative" f'(t) which is a generalized function: f'(t) = f'_r(t) + f'_s(t) . The regular part is the ordinary derivative of f(t) (except at the break points, where it is undefined). The singular part is a sum of delta functions, one for each break in the graph:

(f(a+)-f(a-)) delta(t-a)

There's no separate notation for the generalized derivative to distinguish it from the ordinary derivative, and we will just write f'(t) or dot-x (t).

For example, if f(t) = t + 2 u(t) ,
f'(t) = 1 + 2 delta(t)
f'_r(t) = 1
f'_s(t) = 2 delta(t)

Conversely, if g(t) is a generalized function then

f(t) = int_a^t g(t) dt

makes sense: take the cumulative total of the regular part by integrating as usual, and add in whatever contributions the delta functions make.

Then FTC is still true: f(b) - f(a) = int_a^b f'(t) dt

[2] Let's solve some differential equations with discontinuous input signals. In this part of the course we will make the following standing assumptions (which are covered by the general rubric "rest initial conditions":

(1) the input signal is zero for $t < 0$
(2) the desired system response is zero for $t < 0$
(3) the solution has as many derivatives as possible

[3] First order

Ex 1. $x' + kx = r\,u(t)$.

Interpretation: I add uranium to the reactor at a constant rate of r kg/year, say, starting at $t = 0$.

Look for a continuous solution. Thus $x(0) = 0$, and for $t > 0$ we must solve

$$x' + kx = r \; .$$

The general solution is $x(t) = (r/k) + ce^{-kt}$. To find c , we use $x(0) = 0$: $0 = x(0) = (r/k) + c$, $c = -r/k$,

$$x(t) = (r/k) (1 - e^{-kt}) \qquad t > 0 \; .$$

With $r = 1$, this is the *unit step response,* sometimes written $v(t)$. To be more precise, we could write

$$v(t) = u(t) (1/k) (1 - e^{-kt})$$

Question 24.1. For the solution to $v' + kv = u(t)$ with rest initial conditions, $v(0-) = 0$. What is $v'(0+)$?

1. $v'(0+) = 0$
2. $v'(0+) = 1/k$
3. $v'(0+) = 1$
4. $v'(0+) = k$
5. Don't know

Ans: (3): $v(0) = 0$, so $v'(0+) = u(0+) = 1$.

Ex 2. $x' + kx = \delta(t)$.

Interpretation: I insert one kilogram of uranium into the reactor at $t = 0$ and leave it alone thereafter. The corresponding rate is given by the delta function. The solution can't be continuous; $x(0-) = 0$ but $x(0+) = 1$. Thereafter, though, the equation is

$$x' + kx = 0$$

--- homogeneous, with solution e^{-kt} , $t > 0$.

This is the *unit impulse response*, which we will write $w(t)$. In some sense it is the simplest nontrivial solution; you just give the system a unit kick at $t = 0$, stand back, and watch the result. Since we began with rest initial conditions, the full solution is

$$w(t) = u(t) e^{-kt}$$

Question 24.2. For solution $w(t)$ to $x' + kx = \delta(t)$ with rest initial conditions, what is $w'(0+)$?

1. $w'(0+) = 0$
2. $w'(0+) = - 1/k$
3. $w'(0+) = - 1$
4. $w'(0+) = - k$
5. Don't know

Ans: (4) : $w(0+) = 1$, while $\delta(0+) = 0$, so $w'(0+) = -k$.

[4] Second order

Ex 3. $mx'' + kx = u(t)$

Interpretation: at $t = 0$ a steady force starts to act on the mass. The mass does not change position abruptly, nor does it change velocity instantaneously. Therefore we should expect a solution which is continuous with continuous derivative; only the acceleration experiences a discontinuity.

There is a constant solution: $x = 1/k$. So the general solution is

$$x(t) = (1/k) + a \cos(\omega_n t) + b \sin(\omega_n t) .$$

where $\omega_n = \sqrt{k/m}$

$$0 = x(0) = (1/k) + a$$

$$0 = x'(0) = - a \omega_n \sin(\omega_n t) + b \omega_n \cos(\omega_n t)$$

so $a = - 1/k$, $b = 0$,

$$v(t) = u(t) (1/k) (1 - \cos(\omega_n t)$$

Ex 4. $mx'' + kx = \delta(t)$

Interpretation: There is a sudden very brief very intense force, rather like getting hit on the head by a hammer. The effect is to increase the momentum instantaneously, without changing the position.

Since $x(0) = 0$ and x is continuous, $x(t)$ small for small t , so for small t we have approximately $mv' = \delta(t)$, or $v' = (1/m) \delta(t)$, so v increases abruptly by $(1/m)$ at $t = 0$. The momentum increases by 1 . This is a "unit *impulse*."

So for $t > 0$ the equation is equivalent to

$$mx'' + kx = 0 , \quad x(0+) = 0 , x'(0+) = 1/m$$

--- homogeneous. General solution is sinusoidal with circular frequency omega_n = sqrt{k/m} . x(0+) = 0 implies sine:

    x = c sin(omega_n t) ,   x' = c omega_n cos(omega_n t)

so 1/m = c omega_n   or c = 1/(m omega_n)

and the unit impulse response is

    w = u(t) (1/(m omega_n) sin(omega_n t)

Question 24.3. For solution w(t) to mx" + kx = delta(t)
with rest initial conditions, what is w'(0+) ?

1. w'(0+) = 0
2. w'(0+) = omega_n
3. w'(0+) = k
4. w'(0+) = k/m
5. w'(0+) = 1/m
6. Don't know

Ans: (5): As we saw.