18.03 Class 5, Feb 15, 2008

Complex Numbers, roots of unity

[1] Complex algebra
[2] Complex conjugation
[3] Polar multiplication

Complex numbers provide a tool for expressing aspects of the real world.
Anything with an amplitude and a phase is secretly a complex number,
and it is worthwhile bringing that secret identity out in the open.

[1] Complex Algebra

We think of the real numbers as filling out a line.
The complex numbers fill out a plane. The point up one unit
from 0 is written i (but written j by many engineers).
Addition and multiplication by real numbers is as
vectors. The new thing is $i^2 = -1$. The usual rules of algebra apply.
For example FOIL:

     $(1 + i)(1 + 2i) = 1 + 2i + i - 2 = -1 + 3i.$

Every complex number can be written as a + bi with a and b real.

a = Re(a+bi) the real part
b = Im(a+bi) the imaginary part: NB this is a real number.

Maybe complex numbers seem obscure because you are used to imagining
numbers by giving them units: 5 cars, or -3 degrees. Complex numbers do
not accept units. Also, there is no ordering on complex numbers, no "<."

Question 1. Multiplication by i has the following effect
on a complex number.

1. It rotates the number around the origin by 90 degrees counterclockwise.

2. It rotates the number around the origin by 90 degrees clockwise.

3. It takes a number to the number pointing in the opposite direction
with the same distance from the origin.

4. It reflects the number across the imaginary axis.

5. It reflects the number across the real axis.

6. None of the above.

Blank: Don't know.

Compute: $i(a + bi) = -b + ai$,
which is rotated by 90 degrees counterclockwise: Ans: 1.

[2] Complex conjugation

Notice the formula

    (a+bi)(a-bi) = a^2 - (bi)^2 = a^2 + b^2   (*)

(which is the square of the distance to the origin).

This lets us divide by a complex number: For example

    i/(1+2i) = (i/(1+2i)) ((1-2i)/(1-2i))

            = (i+2)/(1+4) = (2+i)/5 .

(*) suggests that we give a special name to the quantity a-bi :
it is the "complex conjugate" of a+bi and is denoted by an overline:

    $\overline{a+bi}$ = a-bi : reflect across the real axis.

Conjugation satisfies $\overline{w+z} = \overline{w} + \overline{z}$, $\overline{wz} = \overline{w}.\overline{z}$, $\overline{\overline{z}} = z$

Proofs: w = a+bi , z = c+di .

    $\overline{w}$ + $\overline{z}$ = (a-bi) + (c-di) = (a+c) - (b+d)i = $\overline{w+z}$

    $\overline{w}.\overline{z}$ = (a-bi)(c-di) = (ac-bd) - (ad+bc)i

    w.z = (ac-bd) + (ad+bc)i

which conjugates to the line above.

Question 2: If $\overline{z}$ = -z , then

1. z is purely imaginary

2. z is real

3. z lies on the unit circle

4. z = 0

5. None of the above

Blank. Don't know

Ans: 1.

[3] Polar multiplication

Being points in the plane, complex numbers have polar descriptions.
The distance of z from zero is

| z| = "absolute value" = "modulus" = "magnitude" of z.

The angle up from the positive real axis is

Arg(z) = "argument" = "angle" of z.   As usual, it's only well defined
                                       up to adding multiples of 2 pi.

For example Arg(1) = 0 , Arg(i) = pi/2 , Arg(-1) = pi.

| z|^2 = z z̄   and tan(Arg(z)) = b/a = Im(z)/Re(z) .

- Magnitudes Multiply : |wz| = |w||z| .

proof: It's not pleasant to compute absolute values - they involve square
roots - but it's easy to compute squares:

| wz|^2 = (wz)(w̄z̄) = wz̄wz̄ = w̄w̄z̄z̄ = |w|^2|z|^2 = (|w||z|)^2

It follows that |wz| = |w||z| since both sides are positive.

For example, the product of complex numbers on the unit circle lies again
on the unit circle. I drew a picture.

- Angles Add: Arg(wz) = Arg(w) + Arg(z)

I'll check this in case w and z are both on the unit circle. Then:

(cos a + i sin a)(cos b + i sin b) =

((cos a)(cos b) - (sin a)(sin b)) + i ((cos a)(sin b) + (sin a)(cos b))

= cos(a+b) + i sin(a+b)

using the angle addition formulas for cos and sin .

In fact multiplication of complex numbers contains in it the angle
addition formulas for sin and cos , and if you understand complex
numbers you'll never have to memorize those formulas again.

This checks with our question about multiplication by i above.
Here's another question about multiplication:

Question 3. (1+i)^4 =

1. - 1

2. 4

3. - 4

4. - sqrt(2)

5. 4 i

6. None of the above

Blank: Don't know

	Modulus	Argument	a+bi
(1+i)^0	1	0	1
(1+i)^1	sqrt(2)	pi/4	1+i
(1+i)^2	2	pi/2	2i
(1+i)^3	2 sqrt(2)	3pi/4	-2+2i
(1+i)^4	4	pio	-4

These powers of (1+i) spiral away from the origin.

The powers all lie on a spiral emanating from the origin. Ans: (3) .
Thus 1 + i is a fourth root of -4.

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