18.03 Class 4, Feb 10, 2010

First order linear equations: integrating factors

[1] First order homogeneous linear equations
[2] Newtonian cooling
[3] Integrating factor (IF)
[4] Particular solution, transient, initial condition
[5] General formula for IF

Definition: A "linear ODE" is one that can be put in the "standard form"

$$\boxed{r(t)x' + p(t)x = q(t)} \qquad x = x(t)$$

r(t), p(t) are the "coefficients" [I may have called q(t) also a coefficient also on Monday; this is not correct, fix it if I did.]

The left hand side represents the "system," and the right hand side arises from an "input signal." A solution x(t) is a "system response" or "output signal."

We can always divide through by r(t), to get an equation of the Reduced standard form:

$$\boxed{x' + p(t)x = q(t)} \qquad x = x(t) \qquad (*)$$

The equation is "homogEneous" if q is the "null signal," q(t) = 0 .
This corresponds to letting the system evolve in isolation:
In the bank example, no deposits and no withdrawals.
In the RC example, the power source is not providing any voltage increase.

The homogeneous linear equation

$$x' + p(t) x = 0 \qquad (*)\_h$$

is separable. Here's the solution,
in general on the left, with an example (with p(t) = 2t) on the right:

	x' + p(t)x = 0	x' + 2tx = 0
Separate:	dx/x = - p(t) dt	dx/x = - 2t dt
Integrate:	ln|x| = - int p(t) dt + c	ln|x| = - t^2 + c
Exponentiate:	|x| = e^c e^{ - int p(t) dt }	|x| = e^c e^{-t^2}

Eliminate the absolute value and reintroduce the lost solution:

$$x = C e^{- \int p(t)\, dt} \qquad x = C e^{-t^2}$$

In the example, we chose a particular anti-derivative of k, namely kt. That is what I really have in mind to do in general. The constant of integration is taken care of by the constant C.

So the general solution to $(*)_h$ has the form $C x_h$, where x_h is *any* nonzero solution:

$$x_h = e^{- \int p(t)\, dt} , \quad x = C x_h$$

We will see that the general case can be solved by an algebraic trick that produces a sequence of two integrations.

[2] Example: Diffusion, e.g. of heat.

About this time of year I start to think about summer. I put my rootbeer in a cooler but it still gets warm. Let's model its temperature by an ODE.

$x(t)$ = root beer temperature at time t.

The greater the temperature difference between inside and outside, the faster $x(t)$ changes.

Simplest ("linear") model of this:

$$x'(t) = k (T_{ext}(t) - x(t))$$

where $T_{ext}(t)$ is the "external" temperature. Sanity check: When $T_{ext}(t) > T(t)$, $x'(t) > 0$ (assuming $k > 0$). We get a linear equation:

$$x' - k x = k T_{ext}$$

This is "Newton's law of cooling." k could depend upon t and we would still have a linear equation, but let's suppose that we are not watching the process for so long that the insulation of the cooler starts to break down!

Systems and signals analysis:
The system is the cooler.
The output signal = system response is $x(t)$, the temperature in the cooler.
The input signal is the external temperature $T_{ext}(t)$.

Note that the right-hand side is k times the input signal, not the input signal itself.

What constitutes the input and output signals is a matter of the interpretation of the equation, not of the equation itself.

Question 4.1: k large means
1. good insulation
2. bad insulation
Blank. don't know.

k is small when the insulation is good, large when it is bad. It's zero when the insulation is perfect. k is a COUPLING CONSTANT When it is zero, the temperature inside the cooler is decoupled from the temperature outside. In the construction industry, a number like k is pasted on windows; it's called the U-value of the window.

Let's take $k = 1/3$, for example.

Suppose the temperature outside is rising at a constant rate: say

$$T_{ext} = 60 + 6t \quad \text{(in hours after 10:00)}$$

and we need an initial condition: let's say $x(0) = 32$.

So the IVP is $x' + (1/3) x = 20 + 2t$, $x(0) = 32$. (cooler)

This isn't separable: it's something new. We'll describe a method which works for ANY first order linear ODE.

[3] Method: Integrating factors (Euler)

This method is based on the product rule for differentiation:

$$(d/dt) (u x) = ux' + u'x$$

For example, suppose we have the equation

$$t x' + 2 x = t$$

(This is not separable; it is linear and in standard form, but not reduced standard form.) Here's a *trick*. Multiply both sides by t :

$$t^2 x' + 2t x = t^2$$

The left hand side is now the derivative of a product:

$$(d/dt) (t^2 x) = t^2$$

We can solve this by integrating:

$$t^2 x = t^3/3 + c$$

so $$x = t/3 + c t^{-2}$$

[In the first lecture, I posed this (with a different righthand side) as a flashcard problem, but I did it just after describing the calculation of an integrating factor for a *reduced* equation. The reduced equation is $x' + 2x/t = 1$, and this has integrating factor t^2 . So it was a poorly placed question.]

That was great! The factor t we multiplied by is an "integrating factor." I guessed it here. Often you can. The factor to use in the cooler equation and other equations may not be so obvious. Here's a calculation, for a linear equation in reduced form,

$$x' + p(t)x = q(t)$$

Multiply both sides by u

$$u x' + p u x = u q$$

In order for the right hand side to be $(d/dt)(ux) = ux' + u'x$,
the function u must satisfy the differential equation

$$u' = p u$$

This is separable, and we'll carry out the separation in general in a minute.
In the cooler equation, the coefficient $p(t)$ is constant.
In that case we have the natural growth equation!

$$u = e^{pt}$$

(I am choosing a value for the constant of integration, because I need just one integrating factor, any one.)

In the case of the cooler problem, $p = 1/3$, so we have:

$$(d/dt) (e^{t/3} x) = (20 + 2t) e^{t/3}$$

Integrate:

$$e^{t/3} x = 60 e^{t/3} + \int 2t e^{t/3} dt$$

Um. Parts: $\int u dv = uv - \int v du$

$$u = 2t \ , \quad dv = e^{t/3} dt$$
$$du = 2dt \ , \quad v = 3 e^{t/3}$$

$$e^{t/3} x = 60 e^{t/3} + 6 t e^{t/3} - 18 e^{t/3} + c$$

$$= (42 + 6 t) e^{t/3} + c$$

Solve for x:

$$x = (42 + 6t) + c e^{-t/3}$$

That's the general solution. Remember, you can check it easily.

u is an "integrating factor."

[4] We still should finish the IVP process:

$$32 = x(0) = 42 + c \quad \text{so} \quad c = -10 :$$

$$x = 42 + 6t - 10 e^{-t/3}$$

We just want one u , not the general u : so the exponent could be any antiderivative of p . In the example, $p = 1/3$ was constant and we took $u = e^{t/3}$.

Note the structure of the genearal solution:

$$x = x_p + c\, u^{-1}$$ where

. x_p is a solution, *any solution*. It's called a PARTICULAR SOLUTION but this is a very poor name, because there is nothing particular about it. I this case we chose one with a pretty simple formula -- $x_p = 42 + 6t$.)

. u is an integrating factor.

Very often x_h approaches zero with time, as this one does. It is then called a TRANSIENT. All solutions come to look more and more alike as time goes on. This is a funnel!

I graphed the solutions $42 + 6t$ and x , and some others along with T_{ext} . If the temperature in the cooler is more than 60 degrees at the start, then it declines at first, crosses the nullcline $x = 60 + 6t$ where it is momentarily in equilibrium with the outside, and then rises to become asymptotic to $42 + 6t$ like every other solution.

[5] Let's compute an integrating factor for the general first order linear equation (*) : we are to solve $u' = up$.

This is a separable equation: $du/u = p\, dt$

$$\ln|u| = \int p\, dt$$

The constant of integration is in the indefinite integral.

$$|u| = e^{\int p\, dt}$$

Now there is a choice of sign. Pick one and go with it; say

$$u = e^{\int p\, dt}$$

That gives you an integrating factor. Any nonzero multiple serves as well.

Note that this is the reciprocal of a solution to the homogeneous equation:

$$u = xh^{-1}$$

This gets fed into the solution for x :

$$x = u^{-1} \int u\, q\, dt$$

and the constant of integration in the integral lets us write

$$x = x_p + c\, x_h$$