Problems: Surface Independence

Suppose that $\mathbf{F} = \nabla \times \mathbf{G}$, where the components of $\mathbf{G}$ have continuous second partial derivatives. Suppose also that S is a closed, positively-oriented surface divided into two parts by a closed curve C. Apply Stokes' theorem to show that $\iint_S \mathbf{F} \cdot \mathbf{n} \, dS = 0$.

Answer: We start by drawing a picture.

By Stokes' theorem, $\oint_C \mathbf{G} \cdot d\mathbf{r} = \iint_{S_1} \text{curl} \mathbf{G} \, dA = \iint_{S_1} \mathbf{F} \cdot \mathbf{n} \, dS$. Similarly,

$$\oint_{-C} \mathbf{G} \cdot d\mathbf{r} = \iint_{S_2} \mathbf{F} \cdot \mathbf{n} \, dS.$$

We sum the two results to get:

$$\iint_S \mathbf{F} \cdot \mathbf{n} \, dS = \iint_{S_1} \mathbf{F} \cdot \mathbf{n} \, dS + \iint_{S_2} \mathbf{F} \cdot \mathbf{n} \, dS = \oint_C \mathbf{G} \cdot d\mathbf{r} + \oint_{-C} \mathbf{G} \cdot d\mathbf{r} = 0.$$

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