CHRISTINE BREINER:

Welcome back to recitation. In this video, I'd like us to consider the following problem.

The first part is I'd like to know, for what values of b is this vector field F conservative? And F is defined as y*i plus the quantity x plus b*y*z j plus the quantity y squared plus 1, k. So as you can see, the only thing we're allowed to manipulate in this problem is b. b will be some real number. And I want to know, what real numbers can I put in for b so that this vector field is conservative.

The second part of this problem is for each b-value you determined from one, find a potential function. So fix the b-value for one of the ones that is acceptable, based on number one, and then find the potential function. And then the third part says that you should explain why F dot dr is exact, and this is obviously for the b-values determined from one.

So the second and third part are once you know the b-values. And you're only going to use those b-values that make F conservative, because that's the place where we can talk about finding a potential function, and that's where we can talk about F dot dr being exact, are exactly those values. OK. So why don't you pause the video, work on these three problems, and then when you're feeling good about them, bring the video back up, I'll show you what I did.

OK, welcome back. Again, we're interested in doing three things with this vector field. The first thing we want to do is to find the values of b that make this vector field conservative. So I will start with that part.

And as we know from the lecture, the thing I ultimately need to do is I need to find the curl of F. OK, so the curl of F is going to measure how far F is from being conservative. So if the curl of F is 0, I'm going to have F being conservative. So that's really what I'm interested in doing first.

So I'm actually just going to rewrite what the curl of F actually is. So I'm going to let F-- I'm going to denote in our usual way by P, Q, R. And in this case, that's specifically y comma x plus b*y*z comma y squared plus 1. OK, that's my P, Q, R. So P is y, Q is x plus b*y*z, and R is y squared plus 1. And now the curl of F-- which was found in the lecture, so I'm not going to show you again how to find it, I'm just going to write the formula for it-- is exactly the following

vector.

It's the derivative of the R-th component with respect to y, minus the Q-th component with respect to z. That's the i-value. The j-value is P sub z minus R sub x, j. The k-th value is Q sub x minus P sub y, k. OK, so there are three components here. And let's just start figuring out what these values are, and then we'll see what kind of restrictions we have on b.

So let's start doing this. So again, this is P, this is Q, and this is R. So R sub y is the derivative of this component with respect to y. That's just 2y. Q sub z is the derivative of this with respect to z. Well, this is 0, and this is b*y. OK.

Let's look at the rest first. P sub z: the derivative of this with respect to z is 0. The derivative of R with respect to x is 0. So that doesn't have any b's in it at all. And then Q sub x minus P sub y. Q is the middle one. Q sub x is 1. P is the first one. P sub y is 1. OK, so what do we get here? I should have written equals there, maybe.

OK, so the j-th component is 0. And the k-th component is 0. So all I'm left with is 2y minus b*y, i. And if I want F to be conservative, this quantity has to be 0, so I see there's only one b-value that's going to work, and that is b is equal to 2. OK?

So I know in part one, the answer to the question is just for b equals 2, is F conservative. That was, maybe, poorly phrased. F is conservative only when b is 2. OK. And that's because the curl of F is 0 only when b is 2. All right. So now we can move on to the second part.

And the second part is for this particular value of b, find a potential function. And our strategy for that is going to be one of the methods from lecture. And it's going to be the method from lecture that in three dimensions is much easier than the other. So the one method in lecture that's easy in three dimensions is where you start at the origin, and you integrate F dot dr along a curve that's made up of line segments. So this strategy I've done before in two dimensions, in one of the problems in recitation. Now, we'll see it in three dimensions.

So what we're going to do is we're going to integrate along a certain curve F dot dr. And this curve is going to go from the origin to (x_1, y_1, z_1). And that will give us f of (x_1, y_1, z_1). OK. So this is a sort of general strategy, and now we'll talk about it specifically. This will actually be f of (x_1, y_1, z_1) plus a constant, but we'll deal with that part right at the end.

OK, so C in this case is going to be made up of three curves. And I'm going to draw them, in a picture, and then we're going to describe them. So I'm going to start at (0, 0, 0). My first curve

will go out to x_1 comma 0, 0, and that's going to be the curve C_1. Oops. I want that to go the other way. That way.

OK, C_1 is going to go from the origin to x_1 comma 0, 0. So the y- and z-values are going to be 0 and 0 all the way along, and the x-value is going to change.

My next one-- I'm going to make it long so I can have enough room to write-- that's going to be my C_2. And that's going to be x_1 comma y_1 comma 0. So in the end, what I've done is I've taken my x-value, I've kept it fixed all the way along here, but I'm varying the y-value out to y_1.

And then the last one is going to go straight up. Right there. And so it's going to be with the x-value and y-value fixed. And at the end, I will be at x_1, y_1, z_1. And this is C3. So those are my three curves-- And this one, I'm going to move in this direction. Those are my three curves.

And I want to point out that in order to understand how to simplify this problem, I'm going to have to remind myself what F dot dr is. OK. So F dot dr is P*dx plus Q*dy plus R*dz. Right? That's what F dot dr is.

And so what I'm interested in, I'm going to integrate each of these things along C_1, C_2, C_3. But let's notice what happens along certain numbers of these curves. If we come back over here.

On C_1, y is fixed and z as fixed. So dy and dz are both 0. So on C1, I only have to integrate P. OK, so I'm going to keep track of that.

On C_1-- which, C_1 is parametrized by x, 0 to x-- I only need to worry about the P. P of x, 0, 0, dx. This is my C_1 component, and there's nothing else there, because these two are both 0. Right?

Now let's consider what happens on C_2. If I look here. On C_2, the x-value is fixed and the z-value is fixed. x is fixed at x_1 and z is fixed at 0. And so dx and dz are both 0, because x and z are not changing. So there's only a dy component.

So on C_2, which is parametrized in y-- from 0 to y_1-- I'm only interested in Q at x_1 comma y comma 0, dy. Again, this component is 0 on C_2 because dx is 0. And this component is 0 on C_2 because dz is 0. And this component-- I'm evaluating it-- x is fixed at x_1, z is fixed at 0, and the y is varying from 0 to y1.

And then there's one more component, and I'm going to write it below, and then we'll do the rest over here. And the third component is the C_3 component. Now, not surprisingly-- if I come back over here-- because x and y are fixed all along the C3 component, the only thing that's changing is z. So dx and dy are 0, so I'm only worried about the dz part. OK.

So again, as happened before, I only had P in the first one and Q in the second one, and now I have R, only, in the third one. And it's parametrized in z, from 0 to z_1. That's what z varies over. And it's going to be R at x_1 comma y_1 comma z dz. Because the x's are fixed at x_1, the y is fixed at y_1, but z is varying from 0 to z_1. All right.

So I have these three parts, and now I just have to fill them in with the vector field that I have. I want to find what P is at (x, 0, 0), what Q is at $(x_1, y_1, 0)$, and what R is at (x_1, y_1, z). And then integrate. So I have two steps left. One is plugging in and one is evaluating. So let me remind us what P, Q, and R actually are, and then we'll see what we get.

Let me write it again. Maybe here. [P, Q, R] was equal to y comma x plus 2y*z-- I'll put it here so you don't have to look and I don't have to look-- and then y squared plus 1. OK.

So P at x comma 0 comma 0. Well, if I plug in 0 for y, P is 0. So P at (x, 0, 0) is equal to 0. So I get nothing to integrate in the first part. That's nice. OK.

Now, what is Q at x_1 comma y comma 0? Well, that would be an x_1 here. 0 for y makes this term go away. So it's just equal to x_1. Right?

And then R at x_1 comma y_1 comma z is y_1 squared plus 1. So now I'm going to substitute these into what I'm integrating.

So in the first one, there's nothing there. Let me just write it right here. The Q is going to be the integral from 0 to y_1 of x_1 dy. And the R part is going to be the integral from 0 to z_1 of y_1 squared plus 1 dz. OK, so the P part was disappeared. This is the Q part evaluated where I needed it to be evaluated. It's just x_1 dy. And the R part evaluated at (x_1, y_1, z) is just y_1 squared plus 1. And so I integrate that in z.

So if I integrate this in y, all I get is x_1*y evaluated 0 and y1. So here I just get an x_1*y_1. Right? And then here, if I integrate this in z, I just get a z-- and so I evaluate that at z_1 and 0-- I just get z_1 times y1 squared plus 1. OK. So this is actually my potential function.

And so let me write it formally. I should actually say, this is my final answer. Right? I was integrating. This is actually what I get. And so what I was trying to find, if you remember-- I'm going to come back here and just mention it again. What I was doing was I was integrating along a curve F dot dr, to give me f of (x_1, y_1, z_1). Right? So now I've found it. The only thing I said is we also have to allow for there to be a constant. OK.

So the potential function is actually exactly this function plus a constant. OK. So this is f of (x_1, y_1, z_1). And since I don't have much room above, I'll just write it below. This is f of x_1, y_1, z_1. So that's my potential function for this vector field, capital F, when it is conservative. So when b is equal to 2.

OK, and there was one last part to this question. Right? So if we come all the way back over, we're reminded of one last part. It was explain why F dot dr is exact for the b values determined from number one. And the reason is exactly because of the following thing.

F is conservative based on the fact that b is 2. And so when we talk about when F dot dr is exact, the simplest case is capital F is conservative, and I'm on a simply connected domain. OK. And if you notice, capital F is defined for all values x, y, z, and is differentiable for all values of x, y, z. So F is defined and differentiable everywhere on R^3. R^3 is simply connected.

So we have a conservative vector field on a simply connected region. And that's what it means. That's one way that we have of knowing F dot dr is exact. And so that actually answers the third part of the question.

So again, let me just remind you what we did. We started with a vector field F, we found values for b that made that vector field conservative, and then we used one of the techniques in class to find a potential function for that value of b. So there were a number of steps involved, but ultimately, again, it's the same type of problem you've seen before, when F was a vector field in two dimensions. So it shouldn't be feeling too different from some of the stuff you saw earlier.

OK, I think that's where I'll stop.