V9.2 Surface Integrals

2. Flux through a cylinder and sphere.

We now show how to calculate the flux integral, beginning with two surfaces where $\mathbf{n}$ and dS are easy to calculate — the cylinder and the sphere.

Example 1. Find the flux of $\mathbf{F} = z\,\mathbf{i} + x\,\mathbf{j} + y\,\mathbf{k}$ outward through the portion of the cylinder $x^2 + y^2 = a^2$ in the first octant and below the plane $z = h$.

Solution. The piece of cylinder is pictured. The word "outward" suggests that we orient the cylinder so that $\mathbf{n}$ points outward, i.e., away from the z-axis. Since by inspection $\mathbf{n}$ is radially outward and horizontal,

$$(5) \qquad\qquad \mathbf{n} = \frac{x\,\mathbf{i} + y\,\mathbf{j}}{a} \ .$$

(This is the outward normal to the circle $x^2 + y^2 = a^2$ in the xy-plane; $\mathbf{n}$ has no z-component since it is horizontal. We divide by a to make its length 1.)
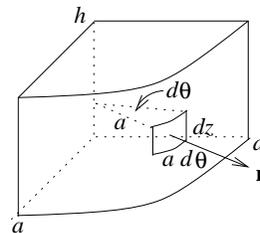
To get dS, the infinitesimal element of surface area, we use cylindrical coordinates to parametrize the cylinder:

$$(6) \qquad\qquad x = a\cos\theta, \qquad y = a\sin\theta \qquad z = z \ .$$

As the parameters θ and z vary, the whole cylinder is traced out ; the piece we want satisfies $0 \le \theta \le \pi/2$, $0 \le z \le h$. The natural way to subdivide the cylinder is to use little pieces of curved rectangle like the one shown, bounded by two horizontal circles and two vertical lines on the surface. Its area dS is the product of its height and width:

$$(7) \qquad\qquad dS = dz \cdot a\,d\theta \ .$$

Having obtained $\mathbf{n}$ and dS, the rest of the work is routine. We express the integrand of our surface integral (3) in terms of z and θ:

$$\begin{aligned}
\mathbf{F}\cdot\mathbf{n}\,dS &= \frac{zx + xy}{a}\cdot a\,dz\,d\theta \ , && \text{by (5) and (7);} \\
&= (az\cos\theta + a^2\sin\theta\cos\theta)\,dz\,d\theta, && \text{using (6).}
\end{aligned}$$

This last step is essential, since the dz and $d\theta$ tell us the surface integral will be calculated in terms of z and θ, and therefore the integrand must use these variables also. We can now calculate the flux through S:

$$\begin{aligned}
\iint_S \mathbf{F}\cdot\mathbf{n}\,dS &= \int_0^{\pi/2}\int_0^h (az\cos\theta + a^2\sin\theta\cos\theta)\,dz\,d\theta \\
\text{inner integral} &= \frac{ah^2}{2}\cos\theta + a^2 h\sin\theta\cos\theta \\
\text{outer integral} &= \left[\frac{ah^2}{2}\sin\theta + a^2 h\frac{\sin^2\theta}{2}\right]_0^{\pi/2} = \frac{ah}{2}(a+h) \ .
\end{aligned}$$

Example 2. Find the flux of $\mathbf{F} = xz\,\mathbf{i} + yz\,\mathbf{j} + z^2\,\mathbf{k}$ outward through that part of the sphere $x^2 + y^2 + z^2 = a^2$ lying in the first octant $(x, y, z, \ge 0)$.

Solution.   Once again, we begin by finding **n** and dS for the sphere. We take the outside of the sphere as the positive side, so **n** points radially outward from the origin; we see by inspection therefore that

(8) $$\mathbf{n} \;=\; \frac{x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}}{a}\;,$$

where we have divided by a to make **n** a unit vector.

To do the integration, we use spherical coordinates ρ, ϕ, θ. On the surface of the sphere, $\rho = a$, so the coordinates are just the two angles ϕ and θ. The area element dS is most easily found using the volume element:

$$dV = \rho^2 \sin\phi\,d\rho\,d\phi\,d\theta = dS \cdot d\rho \;=\; \text{area} \cdot \text{thickness}$$

so that dividing by the thickness $d\rho$ and setting $\rho = a$, we get

(9) $$dS = a^2 \sin\phi\,d\phi\,d\theta.$$
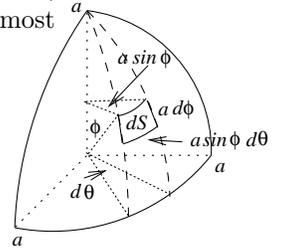
Finally since the area element dS is expressed in terms of ϕ and θ, the integration will be done using these variables, which means we need to express x, y, z in terms of ϕ and θ. We use the formulas expressing Cartesian in terms of spherical coordinates (setting $\rho = a$ since (x, y, z) is on the sphere):

(10) $$x = a \sin\phi \cos\theta, \qquad y = a \sin\phi \sin\theta, \qquad z = a \cos\phi\;.$$

We can now calculate the flux integral (3). By (8) and (9), the integrand is

$$\mathbf{F} \cdot \mathbf{n}\,dS \;=\; \frac{1}{a}\,(x^2 z + y^2 z + z^2 z) \cdot a^2 \sin\phi\,d\phi\,d\theta\;.$$

Using (10), and noting that $x^2 + y^2 + z^2 = a^2$, the integral becomes

$$\iint_S \mathbf{F} \cdot \mathbf{n}\,dS \;=\; a^4 \int_0^{\pi/2} \int_0^{\pi/2} \cos\phi \sin\phi\,d\phi\,d\theta$$

$$= a^4 \frac{\pi}{2}\frac{1}{2}\,\sin^2\phi\,\Big]_0^{\pi/2} \;=\; \frac{\pi a^4}{4}\;.$$

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