**JOEL LEWIS:**   Hi. Welcome back to recitation. In lecture, you've been learning about triple integration, and I have a nice average value problem for you using triple integration here.

So what I'd like you to do is to consider the tetrahedron that has vertices at the origin, and at the points (1, 0, 0) and (0, 1, 0) and (0, 0, 1). So that's one point on each of the positive axes, at distance 1 from the origin. So I've taken the liberty of drawing it here for you. So consider that solid tetrahedron. And what I'd like you to do is find the average distance of the points in that tetrahedron from the xy-plane. All right.

So I'd like you to compute the average value of the distance, as the point ranges over the whole tetrahedron, of its distance from the xy-plane. So why don't you pause the video, spend some time working that out, come back, and we can work it out together.

Hopefully, you had some luck with this problem. Let's get started.

So the average value of a function F over a region R is going to be 1 over the volume of the region times the triple integral over your whole region R of the function value f of x, y, z with respect to volume, in any order you want. So I guess I'll write dV here, and then to evaluate this, you set it up as an iterated integral.

So in our case, the function we're trying to find the average value of is the distance between a point and the xy-plane. But that's an easy function, right? That's just z. For any point in space, its distance from the xy-plane is just its height. It's z-value. So the function that we're seeking to find the average value of is z, and so most of the work of this problem then is going to be in figuring out what the bounds are and then doing the actual integral after that. So OK.

So let's think about the bounds. This tetrahedron is a nice, reasonably simple, geometric object. So in fact, it doesn't matter too much which order you take your bounds. So I think I'm going to do it in the order dz dy dx. You know, I'll do z first, and then y, and then x.

But it doesn't matter. If you do it a different way, it'll probably work out very similarly overall, and you'll still be able to compare the overall process. So let's think about z. Yeah?

So in this tetrahedron, we want to integrate with respect to z first. So we look at this tetrahedron and we say, OK, at a point x and y-- you know, when we choose the x- and y-values, what's the lowest surface-- what's the smallest value z can take-- and what's the upper

surface-- what's the largest value z can take?

So the lowest surface here is the xy-plane. That's the bottom face of this tetrahedron. And for any choice of x and y, the lowest value z can take is when it's in the xy-plane. So when it's equal to 0.

So this is going to be equal, so in our case, so let's set this up. So it's going to be an iterated integral. The function we're integrating is z, and I said we'll do dz dy dx. And we just said that the lowest value that z takes-- the dz-- is 0.

So the highest value that z takes is when it hits this top surface. This plane that passes through the point (1, 0, 0), (0, 1, 0), and (0, 0, 1). All right. So we need to know the equation of that plane. Luckily, that's a pretty easy plane to write down the equation for.

So this slanted plane passing through the three vertices other than the origin is the plane x plus y plus z equals 1. All right. So it's a nice, easy plane to work with. And so what we want to know is what's the value of z on that plane. So we isolate z on one side and we bring everything else over. So we have that top value of z, the largest value of z that z can take when x and y are fixed, is 1 minus x minus 1. So that's what goes up here. So that's the biggest value z can take. OK, good.

So now we need to figure out what the bounds on y are in terms of x. So what I like to do in this case is I like to draw a projection of your surface. So then you're in a two-dimensional world, and then you can look at that image more easily.

So what we're going to do is we're going to look at this tetrahedron and we're going to imagine projecting it down into the xy-plane. So for every point in the tetrahedron, we're going to draw a dot below it in the xy-plane. And then we're going to look at that set of dots. So what that's going to give us is this bottom face of the tetrahedron. Every point of the tetrahedron is above its bottom face. That's not true for every tetrahedron, but it's true for this one. So the region that we're interested in is that bottom face.

So I'm going to draw another picture of it over on my left here. So that region is the region that has vertices (0, 0), and (1, 0), and (0, 1). So it's this triangle. And this bottom edge of the triangle is the line y equals 0. This left edge is the line x equals 0. And this slanted line is the line x plus y equals 1. OK, so this is that same bottom face that we just drew.

But now I've changed my axes to our usual two-dimensional direction with x to the right and y

up. OK. So we're doing dy next. So we need to figure out for a fixed value of x, what are the bounds on y? So we see from this picture that for any fixed value of x, y goes from 0-- the x-axis-- up to this line. OK? So on the x-axis, y takes the value 0, and on this line, y takes the value 1 minus x.

Finally, our outermost variable of integration is x, and so we need to know what are the absolute largest and smallest values that x takes. So we can do that looking either at this picture or looking at our original picture. Either way, it's not hard to see that x just goes between 0 and 1. The smallest value that x takes in this tetrahedron is 0. The largest value it takes is 1.

So that's our integral that we have to compute. So that's not that bad at all. So now you have to go through and you have to actually integrate this. Yeah? And so I'm going to look at my notes just to make sure I don't make any big arithmetic mistakes. So let's see. Now we do these one at a time.

So this innermost integral is an integral of z dz. OK, well that's easy. That's z squared over 2. And then we're taking z squared over 2 between 0 and 1 minus x minus y.

So this innermost integral is z squared over 2 between 0 and 1 minus x minus y. So that's equal to-- so the innermost integral gives us 1 minus x minus y squared over 2. So that's what we get for the innermost integral.

So our integral that we're looking at, then, is equal to the integral, as x goes from 0 to 1, of the integral as y goes from 0 to 1 minus x of this integrand.

So this is the inner one. Let me write that, "inner." That's what I've got here. Just integrating z with respect to z gives me z squared over 2. And then I evaluated at the bounds of the integral.

OK, so now I need to do the middle one. So let's do that up here. So I need to compute the integral. So now I take the bounds, so the middle one is y, and the bounds are from 0 to 1 minus x, of the inner integral. This thing that I just computed. So that's of 1 minus x minus y squared over 2, dy. OK. So All right.

So this isn't that bad. This is a quadratic polynomial in y. And so it's not terribly hard to see. I'm running a little bit out of board space. So I'm not going to give you a full, detailed explanation.

But it's not hard to see, I think, that this integral of 1 minus x minus y squared over 2 with respect to y is 1 minus x minus y cubed over 3, but it's negative, because the sign here is negative. And you could check by differentiating this and seeing that you get that.

And so we have to evaluate that as y goes from 0 to 1 minus x. So what do we get? Well, when y is equal to 1 minus x, this is 0. So we get 0 minus-- and when y is equal to 0, this is minus 1 minus x quantity cubed over 6-- so it's minus minus 1 minus x cubed over 6, so that's just 1 minus x cubed over 6.

And so finally, the outermost integral is we take the inner two integrals and we integrate them with respect to x as x goes from 0 to 1. So it's the integral from 0 to 1 of 1 minus x cubed over 6, dx. And that's going to equal-- I've run out of space-- 1 over 24. All right.

Except I've done something wrong right at the beginning. I hope you all caught me. Right? I had this 1 over V factor here, and it disappeared. Right? I forgot about this 1 over V, so over here, I should have written 1 over V right in front of that integral.

I've correctly computed 1 over 24 as the value of my triple integral, but the average height here isn't 1 over 24. It's 1 over 24V. All right. So the average height is 1 over 24V, and so we need to go and we need to look at our tetrahedron and figure out what its volume is.

So if we come over here and see our tetrahedron. Now this is nice, simple tetrahedron. The volume of a tetrahedron is 1/3 the area of the base times the height, right? So this is a nice, easy tetrahedron. Its height is 1. Its base is a right triangle whose legs are both 1. So the base area is 1/2, so the volume is 1/6. So if the volume is 1/6, and we said the average value is 1 over 24V, so that works out to 1 over 4. So let me write that just in this space. So the average height then is 1 over 4. So that's going to be our final answer.

OK, so let's just recap briefly what we did. We had an average value problem that we started with. So we use this general formula for average value problems. When you have a function f that you want to take its average value of over a region R, what do you do? Well, you take 1 over the volume of the region times the triple integral of your function f with respect to volume over that region. OK.

So this is the average value in general. In our particular case, the function was the height. It was z. And then you have to set it out choosing the proper bounds for your integrals. So in this case, you choose some order of integration based on the region. In this particular case, it's a

nice, simple region. It doesn't matter too much what order you choose. So I chose dz dy dx.

And then what does that mean? So for the innermost one, you look at your original solid. So I'm going to go back and look at this picture again. So for your innermost variable you say-- so if it's z, you say, so when I fix x and y, what's the bottom surface and what's the top surface when I solve that for z in terms of x and y? So here, that was the plane z equals 0, and the plane z equals 1 minus x minus y. So that explains my bounds over here. Why they were 0 and 1 minus x minus y.

Then, when you go to your next variable-- in this case, it was y-- what do you do? Well, first you project to eliminate this first variable. So you project your region down. Down in this case, because it's z. So you project in the z-direction.

And you draw this shadow of your region. So this is what I drew here. This is the shadow of my region in the xy-plane, after I projected it. And then you do the same thing. So now this just like what you did when you had to find bounds for double integrals, when you wrote them as iterated integrals a few lectures ago. And so then you do the same thing. So then, in this case, I was next integrating with respect to y. So I needed to find the bounds on y with respect to x. So I needed to look, in this picture, at the bottom edge and the top edge of this region.

And then your outermost variable, you look at its absolute bounds. So the largest and smallest value it takes on the region. OK, then you have an iterated integral and you evaluate it by successive integrations. OK. So that was what we did. We just did the three integrals, starting from the inside and working our way out.