Identifying Gradient Fields and Exact Differentials

1. Determine whether each of the vector fields below is conservative.

a) $\mathbf{F} = \langle xe^x + y, x \rangle$

b) $\mathbf{F} = \langle xe^x + y, x + 2 \rangle$

c) $\mathbf{F} = \langle xe^x + y + x, x \rangle$

Answer: We know from lecture that if $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$ is continuously differentiable for all x and y, then

$$M_y = N_x \text{ for all } x \text{ and } y \implies \mathbf{F} \text{ is conservative.}$$

Each of the fields in question is continuously differentiable for all x and y.

a) $M = xe^x + y$, $N = x$. $M_y = 1, N_x = 1$. The field is conservative.

b) $M = xe^x + y$, $N = x + 2$. $M_y = 1, N_x = 1$. The field is conservative.

c) $M = xe^x + y + x$, $N = x$. $M_y = 1, N_x = 1$. The field is conservative.

In fact, we can add any function of x to M and any function of y to N without affecting M_y and N_x.

2. Show $(xe^x + y)\,dx + x\,dy$ is exact.

Answer: We know from lecture that $M\,dx + N\,dy$ is an exact differential if and only if $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$ is a gradient field. To show $\mathbf{F}$ is a gradient field, we must show that $\mathbf{F}$ is continuously differentiable and $M_y = N_x$ for all x, y.

Indeed, $\mathbf{F}$ is continuously differentiable for all x, y by inspection. Here $M = xe^x + y$ and $N = x$, so $M_y = N_x = 1$. We conclude that $(xe^x + y)\,dx + x\,dy$ is exact.

3. Compute the two dimensional curl of $\mathbf{F}$ for each of the vector fields below.

a) $\mathbf{F} = \langle x, xe^x + y \rangle$

b) $\mathbf{F} = \mathbf{i} + \mathbf{j}$

c) $\mathbf{F} = \langle xy^2, x^2y \rangle$

Answer: We know that if $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$ then $\text{curl}\mathbf{F} = N_x - M_y$.

a) $\text{curl}\mathbf{F} = (e^x + xe^x) - 0 = e^x(1 + x)$.

(This looks similar to the conservative vector fields from previous problems, but its components have been swapped.)

b) $M = N = 1$ so $\text{curl}\mathbf{F} = 0 - 0 = 0$.

c) $N_x = 2xy$ and $M_y = 2yx$, so $\text{curl}\mathbf{F} = 0$.

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