Work and line integrals

Line integrals: (also called *path integrals*)

Ingredients:

    Field $\mathbf{F} = M\mathbf{i} + N\mathbf{j} = \langle M, N \rangle$

    Curve C: $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} = \langle x, y \rangle \implies d\mathbf{r} = \langle dx, dy \rangle.$
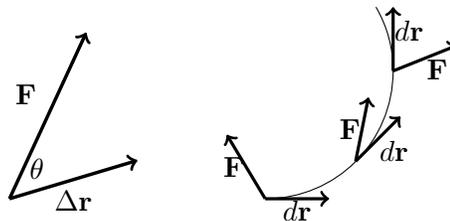
Line integral:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \langle M, N \rangle \cdot \langle dx, dy \rangle = \int_C M \, dx + N \, dy.$$

We need to discuss:

a) What this notation means and how line integrals arise.

b) How to compute them.

c) Their properties and notation.

a) **How line integrals arise.**

The figure on the left shows a force $\mathbf{F}$ being applied over a displacement $\boldsymbol{\Delta}\mathbf{r}$. Work is force times distance, but only the component of the force in the direction of the displacement does any work. So, work $= |\mathbf{F}| \cos\theta \, |\boldsymbol{\Delta}\mathbf{r}| = \mathbf{F} \cdot \boldsymbol{\Delta}\mathbf{r}.$

For a variable force applied over a curve the total work is found by 'summing' the infinitesimal pieces. We call this a line integral and denote it

$$\int_C \mathbf{F} \cdot d\mathbf{r}.$$

It is illustrated in the above figure on the right.

b) **Computing line integrals.**

We show this by steps by example.

Example 1:   Evaluate $I = \displaystyle\int_C x^2 y \, dx + (x - 2y) \, dy$

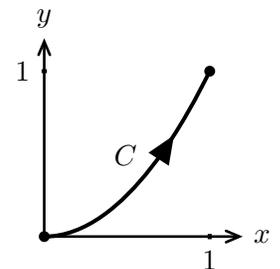over the part of the parabola $y = x^2$ from $(0,0)$ to $(1,1)$.

First, parametrize the curve:

$$x = t, \ y = t^2, \quad 0 \le t \le 1.$$

Note, we specified the range of t to get exactly the part of the curve we wanted.

Next, compute the differentials of x and y:

$$dx = dt, \quad dy = 2t \, dt.$$

Finally substitute everything in the integral and compute the standard single variable integral:
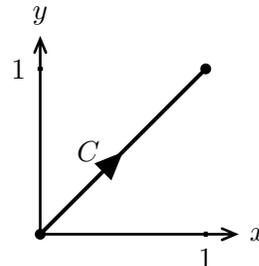
$$I = \int_0^1 t^2(t^2)\, dt + (t - 2t^2)2t\, dt = \int_0^1 t^4 + 2t^2 - 4t^3\, dt = -\frac{2}{15}.$$

Example 2:   (Line integrals depend on the path.)

Same integral as previous example except C is the

straight line from $(0,0)$ to $(1,1)$.

Parametrize curve: $x = t,\ \ y = t,\ \ 0 \le t \le 1. \ \Rightarrow\ dx = dt,\ dy = dt$

$$\Rightarrow\ I = \int_0^1 t^2 \cdot t\, dt + (t - 2t)\, dt = \int_0^1 t^3 - t\, dt = -\frac{1}{4}.$$

Note: this is a different value from example 1 and illustrates the very important fact that, in general, the line integral depends on the path. Later we will learn how to spot the cases when the line integral will be independent of path.

Example 3:   (Line integrals are independent of the parametrization.)

Here we do the same integral as in example 1 except use a different parametrization of C.

Parametrize C: $x = \sin t,\ \ y = \sin^2 t,\ \ 0 \le t \le \pi/2 \ \Rightarrow\ dx = \cos t\, dt,\ dy = 2\sin t \cos t\, dt.$

$$
\begin{aligned}
\Rightarrow\ I\ &=\ \int_0^{\pi/2} \sin^4 t \cos t\, dt + (\sin t - 2\sin^2 t)2\sin t \cos t\, dt \\
&=\ \int_0^{\pi/2} (\sin^4 t + 2\sin^2 t - 4\sin^3 t)\cos t\, dt = \left. \frac{\sin^5 t}{5} + \frac{2}{3}\sin^3 t - \sin^4 t \right|_0^{\pi/2} = -\frac{2}{15}.
\end{aligned}
$$

This is same value as example 1 and illustrates the very important point that the line integral is independent of how the curve is parametrized.

c) **Properties and notation of line integrals**

1. They are independent of parametrization.

2. If you reverse direction on curve then the line integral changes sign. That is,

$$\int_{-C} \mathbf{F} \cdot d\mathbf{r} = - \int_C \mathbf{F} \cdot d\mathbf{r}.$$

$(-C\ \ $ means the same curve traversed in the opposite direction.)

3. If C is closed we use the notation $\oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C M\, dx + N\, dy.$

Example 4: Evaluate $I = \int_C y\,dx + (x + 2y)\,dy$ where C is the curve shown.

Answer: The curve has two pieces so the integral will also

$$I = \int_{C_1} y\,dx + (x + 2y)\,dy + \int_{C_2} y\,dx + (x + 2y)\,dy.$$

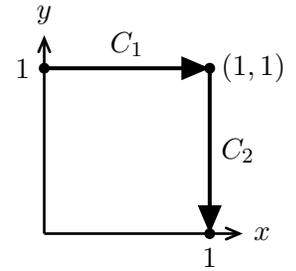Here we see that we don't always need to introduce a new variable t.

$C_1: \ y = 1$, use x as parameter. $0 \le x \le 1 \ \Rightarrow \ dx = dx,\ dy = 0.$

Substituting in the integral $\Rightarrow \int_{C_1} y\,dx + (x + 2y)\,dy = \int_0^1 dx = 1.$

$C_2: \ x = 1$, use y as parameter. y goes from 1 to 0 and $dx = 0$

$\Rightarrow \int_{C_2} y\,dx + (x + 2y)\,dy = \int_1^0 (1 + 2y)\,dy = -\int_0^1 1 + 2y\,dy = -2.$

We get $I = 1 - 2 = -1.$

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