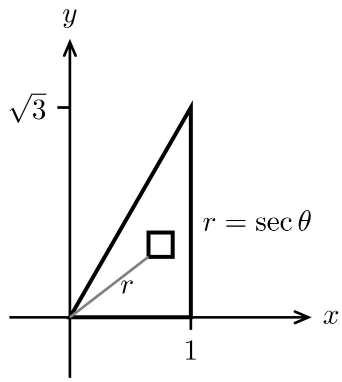
Moment of inertia

1. Let R be the triangle with vertices $(0,0)$, $(1,0)$, $(1,\sqrt{3})$ and density $\delta = 1$. Find the polar moment of inertia.

<u>Answer:</u> The region R is a 30, 60 , 90 triangle.



The polar moment of inertia is the moment of inertia around the origin (that is, the z-axis). The figure shows the triangle and a small square piece within R. If the piece has area dA then its polar moment of inertia is $dI = r^2\delta\, dA$. Summing the contributions of all such pieces and using $\delta = 1$, $dA = r\,dr\,d\theta$, we get the total moment of inertia is

$$I = \iint_R r^2\delta\, dA = \iint_R r^2\, r\, dr\, d\theta = \iint_R r^3\, dr\, d\theta.$$

Next we find the limits of integration in polar coordinates. The line

$$x = 1 \;\Leftrightarrow\; r\cos\theta = 1 \;\Leftrightarrow\; r = \sec\theta.$$

So, using radial stripes, the limits are: (inner) r from 0 to $\sec\theta$; (outer) θ from 0 to $\pi/3$.

Thus,

$$I = \int_0^{\pi/3}\int_0^{\sec\theta} r^3\, dr\, d\theta.$$

Inner integral: $\dfrac{\sec^4\theta}{4}$.

Outer integral: Use $\sec^4\theta = \sec^2\theta\sec^2\theta = (1+\tan^2\theta)\, d(\tan\theta) \;\Rightarrow\;$ the outer integral is

$$\frac{1}{4}\left(\tan\theta + \frac{\tan^3\theta}{3}\right)\Bigg|_0^{\pi/3} = \frac{1}{4}\left(\sqrt{3} + \frac{(\sqrt{3})^3}{3}\right) = \frac{\sqrt{3}}{2}.$$

The polar moment of inertia is $\dfrac{\sqrt{3}}{2}$.

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