# Double integration

1. Set up a double integral for the mass of the planar region R with a variable density $\delta(x, y)$. Use a simple sketch to illustrate the setup.

Note: the problem as stated is abstract, so your solution will also be abstract. The goal here is to focus on the simple idea of slicing the region into small pieces and summing the contributions of each of the pieces that is the basis of integration.

Answer: First note: since this is a planar region the density is in mass/unit area.
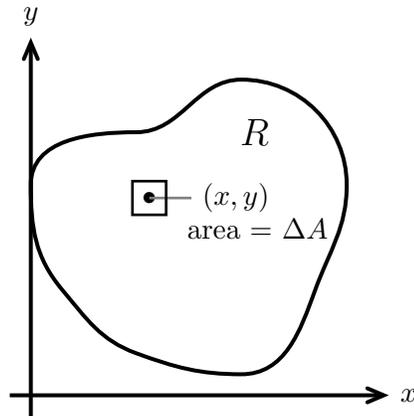
Over a small piece the density (which we assume is continuous) won't vary very much. Thus, if Δm is the mass of the small piece shown in the figure, we have

$$\Delta m \approx \delta(x, y)\Delta A,$$

where (x, y) is a point in the piece and ΔA is the area of the piece.

Now, imagine we slice the region into small pieces and find the mass of each piece, then the total mass of the region is just the sum of the masses of the pieces.

If we let $\Delta A \to 0$ the approximation will become better and the sum will become an integral.

$$M = \int\int_R \delta(x, y)\, dA.$$

If we want to do this more formally, we would slice the region into n pieces and label each piece with and index from 1 to n. If Δm_i is the mass of the i$^{\text{th}}$ piece then

$$\Delta m_i \approx \delta(x_i, y_i)\Delta A_i,$$

where (x_i, y_i) is an arbitrary point in the the piece and ΔA_i is its area. Summing the masses of all the pieces we get

$$\text{Mass of } R \; = \sum_{i=1}^{n}\Delta m_i \approx \sum_{i=1}^{n}\delta(x_i, y_i)\Delta A_i.$$

This is called a Riemann sum. In the limit as n goes to infinity and the area of each piece goes to 0, the sum becomes the integral.

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