Gradient: proof that it is perpendicular to level curves and surfaces

Let $w = f(x, y, z)$ be a function of 3 variables. We will show that at any point $P = (x_0, y_0, z_0)$ on the level surface $f(x, y, z) = c$ (so $f(x_0, y_0, z_0) = c$) the gradient $\nabla f|_P$ is perpendicular to the surface.

By this we mean it is perpendicular to the tangent to any curve that lies on the surface and goes through P. (See figure.)

This follows easily from the chain rule: Let

$$\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$$

be a curve on the level surface with $\mathbf{r}(t_0) = \langle x_0, y_0, z_0\rangle$. We let $g(t) = f(x(t), y(t), z(t))$. Since the curve is on the level surface we have $g(t) = f(x(t), y(t), z(t)) = c$. Differentiating this equation with respect to t gives

$$\frac{dg}{dt} = \left.\frac{\partial f}{\partial x}\right|_P \left.\frac{dx}{dt}\right|_{t_0} + \left.\frac{\partial f}{\partial y}\right|_P \left.\frac{dy}{dt}\right|_{t_0} + \left.\frac{\partial f}{\partial z}\right|_P \left.\frac{dz}{dt}\right|_{t_0} = 0.$$

In vector form this is

$$\begin{aligned} & \left\langle \left.\frac{\partial f}{\partial x}\right|_P, \left.\frac{\partial f}{\partial y}\right|_P, \left.\frac{\partial f}{\partial z}\right|_P \right\rangle \cdot \left\langle \left.\frac{dx}{dt}\right|_{t_0}, \left.\frac{dy}{dt}\right|_{t_0}, \left.\frac{dz}{dt}\right|_{t_0} \right\rangle = 0 \\ \Leftrightarrow \quad & \nabla f|_P \cdot \mathbf{r}'(t_0) = 0. \end{aligned}$$

Since the dot product is 0, we have shown that the gradient is perpendicular to the tangent to any curve that lies on the level surface, which is exactly what we needed to show.