# Critical Points

Critical points:
A standard question in calculus, with applications to many fields, is to find the points where a function reaches its relative maxima and minima.

Just as in single variable calculus we will look for maxima and minima (collectively called *extrema*) at points (x_0, y_0) where the first derivatives are 0. Accordingly we define a *critical point* as any point (x_0, y_0) where
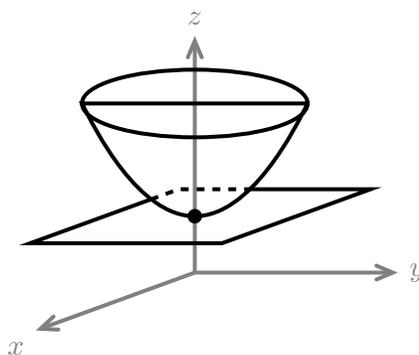
$$\frac{\partial f}{\partial x}(x_0, y_0) = 0 \text{ and } \frac{\partial f}{\partial y}(x_0, y_0) = 0.$$

Often we will abbreviate this as $f_x = 0$ and $f_y = 0$.

Our first job is to verify that relative maxima and minima occur at critical points. The figures below illustrates that they occur at places where the tangent plane is horizontal.
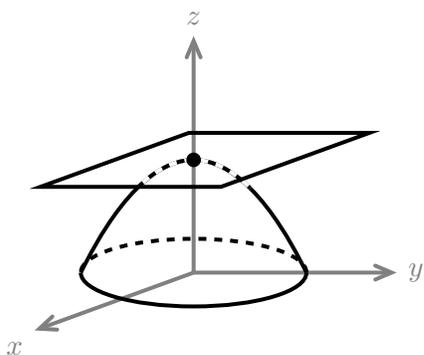
Max. with horizontal tang. plane   Min. with horizontal tang. plane

Since horizontal planes are of the form $z = $ constant. and the equation of the tangent plane at (x_0, y_0, z_0) is

$$z = z_0 + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

we see it is horizontal when

$$f_x(x_0, y_0) = 0 \text{ and } f_y(x_0, y_0) = 0.$$

Thus, extrema occur at critical points. But, just as in single variable calculus, not all critical points are extrema.

**Example:** Find the critical points of $z = x^2 + y^2 + .5$.

Answer: $\dfrac{\partial z}{\partial x} = 2x$ and $\dfrac{\partial z}{\partial y} = 2y$. Clearly the only point where both derivatives are 0 is $(0, 0)$. Thus, there is a single critical point at $(0, 0)$. The figure shows it is clearly the point where z reaches a minimum value. (See the figure above on the right.)

Example: Find the critical points of $z = 1 - x^2 - y^2$.

Answer: $\dfrac{\partial z}{\partial x} = -2x$ and $\dfrac{\partial z}{\partial y} = -2y$. Clearly the only point where both derivatives are 0 is $(0, 0)$. Thus, there is a single critical point at $(0, 0)$. The figure shows it is clearly the point where z reaches a maximum value. (See the figure above on the left.)

Example: Find the critical points of $z = -x^2 + y^2$.

Answer: $\dfrac{\partial z}{\partial x} = -2x$ and $\dfrac{\partial z}{\partial y} = 2y$. Clearly the only point where both derivatives are 0 is $(0,0)$. Thus, there is a single critical point at $(0,0)$. The figure shows it is neither a minimum or a maximum.
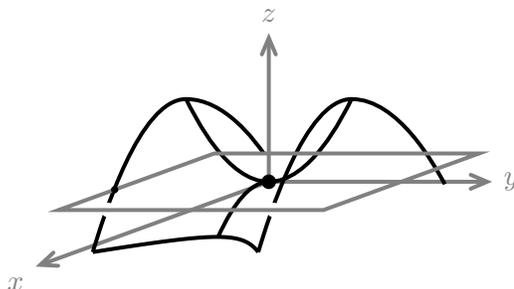
Saddle with horizontal tang. plane

Example: Making a box with minimum material.

A box is made of cardboard with double thick sides, a triple thick bottom, single thick front and back and no top. It's volume = 3.

What dimensions use the least amount of cardboard?

Answer: The box shown has dimensions x, y, and z.

The area of one side $= yz$. There are two double thick sides $\Rightarrow$ cardboard used $= 4yz$.

The area of the front (and back) $= xz$. It is single thick $\Rightarrow$ cardboard used $= 2xz$.

The area of the bottom $= xy$. It is triple thick $\Rightarrow$ cardboard used $= 3xy$.

Thus, the total cardboard used is

$$w = 4yz + 2xz + 3xy.$$

The volume $= 3 = xyz \Rightarrow z = \frac{3}{xy}$. Substituting this in the formula for w gives

$$w = \frac{12}{x} + \frac{6}{y} + 3xy.$$

We find the critical points of w.

$$w_x = -\frac{12}{x^2} + 3y = 0, \quad w_y = -\frac{6}{y^2} + 3x = 0.$$

The first equation implies $y = \dfrac{4}{x^2}$. Substituting this in the second equation gives $-\dfrac{6}{16}x^4 + 3x = 0$. Thus, $x = 0$ or 2. We reject 0 since then y is undefined. Using $x = 2$ we find $y = 1$. Thus, there there is one critical point at $(2,1)$. and at this point we have $z = 3/2$.

This point gives the box with minimum cardboard used because physically we know it must have a minimum somewhere. Later we will learn to check this with the second derivative test.
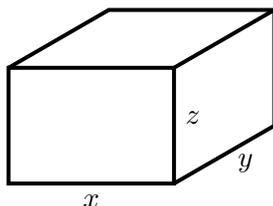
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