Velocity, speed and arc length

Speed

Velocity, being a vector, has a magnitude and a direction. The direction is tangent to the curve traced out by $\mathbf{r}(t)$. The magnitude of its velocity is the speed.

$$\text{speed } = |\mathbf{v}| = \left|\frac{d\mathbf{r}}{dt}\right|.$$

Speed is in units of distance per unit time. It reflects how fast our moving point is moving.

Example: A point goes one time around a circle of radius 1 unit in 3 seconds. What is its average velocity and average speed.

<u>Answer:</u> The distance the point traveled equals the circumference of the circle, 2π. Its net displacement is $\mathbf{0}$, since it ends where it started. Thus, its average speed = distance/time = $2\pi/3$ and its average velocity = displacement/time = $\mathbf{0}$.

If you look carefully, we've used a boldface $\mathbf{0}$ because velocity is a vector.

Our usual symbol for distance traveled is s. For a point moving along a curve the distance traveled is the length of the curve. Because of this we also refer to s as *arc length.*

Notation and nomenclature summary:

Since we will use a variety of notations, we'll collect them here. The unit tangent vector will be explained below. As you should expect, we will also be able to view everything from a geometric perspective.

$\mathbf{r}(t)$ = position.

In the plane $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} = \langle x, y\rangle$

In space $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$.

$\dfrac{d\mathbf{r}}{dt} = \mathbf{v}(t)$ = velocity = tangent vector.

In the plane $\mathbf{v} = x'(t)\mathbf{i} + y'(t)\mathbf{j} = \langle x', y'\rangle$

In space $\mathbf{v} = x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k} = \langle x', y', z'\rangle$.

$\mathbf{T} = \dfrac{\mathbf{v}}{|\mathbf{v}|}$ = unit tangent vector.

s = arclength, speed $= \dfrac{ds}{dt} = |\mathbf{v}|$.

In the plane $\dfrac{ds}{dt} = \sqrt{(x')^2 + (y')^2}$.

In space $\dfrac{ds}{dt} = \sqrt{(x')^2 + (y')^2 + (z')^2}$.

$\mathbf{v} = \dfrac{ds}{dt}\mathbf{T}, \quad \mathbf{T} = \dfrac{\mathbf{v}}{ds/dt}$

$\mathbf{a}(t) = \dfrac{d\mathbf{v}}{dt} = \dfrac{d^2\mathbf{r}}{dt^2}$ = acceleration.

In the plane $\mathbf{a}(t) = x''(t)\mathbf{i} + y''(t)\mathbf{j} = \langle x'', y''\rangle$

In space $\mathbf{a} = x''(t)\mathbf{i} + y''(t)\mathbf{j} + z''(t)\mathbf{k} = \langle x'', y'', z''\rangle$.

Unit tangent vector

As its name implies, the *unit tangent vector* is a unit vector in the same direction as the tangent vector. We usually denote it $\mathbf{T}$. We compute it by dividing the tangent vector by its length. Here are several ways of writing this.

$$\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{d\mathbf{r}/dt}{ds/dt} = \frac{\mathbf{v}}{ds/dt}.$$

Multiply $\mathbf{T}$ by ds/dt gives the formula

$$\mathbf{v} = \frac{ds}{dt}\mathbf{T},$$

which expresses velocity as a magnitute, ds/dt and a direction $\mathbf{T}$.

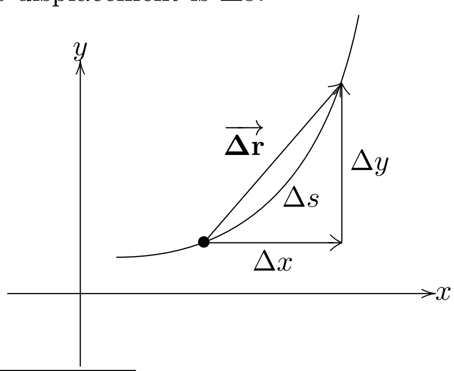
Geometric considerations

Here we'll offer a mathematical justification for our statement that

$$\text{speed } = \frac{ds}{dt} = |\mathbf{v}|.$$

We'll work in two dimensions. The extension to 3D is straightforward.

The figure below shows a curve, and a small displacement $\Delta\mathbf{r}$. The length along the curve from the start to end of the displacement is Δs.



We see $\Delta s \approx |\Delta\mathbf{r}| = \sqrt{(\Delta x)^2 + (\Delta y)^2}$. Dividing by Δt gives

$$\frac{\Delta s}{\Delta t} \approx \left|\frac{\Delta\mathbf{r}}{\Delta t}\right| = \sqrt{\left(\frac{\Delta x}{\Delta t}\right)^2 + \left(\frac{\Delta y}{\Delta t}\right)^2}$$

Taking the limit as $\Delta t \to 0$ gives

$$\frac{ds}{dt} = \left|\frac{d\mathbf{r}}{dt}\right| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}.$$