18.02 Problem Set 1, Part II Solutions

1. There are several ways to set up the tetrahedron for this problem. The simplest way is to inscribe it in the unit cube, so that it has vertices $(0,0,0)$, $(1,1,0)$, $(0,1,1),(1,0,1)$. If you didn't think of this, you could also set it up with one face defined by $(0,0,0),(1,0,0),(0,\sqrt{3}/2,0)$ and then figure out where the fourth vertex should be. In this solution we'll use the simpler inscribed tetrahedron. In this case, we define the plane $\mathcal{P}_1$ to contain $(0,0,0),(1,1,0)$ and $(1,0,1)$; taking the cross product we see a normal to this plane is

$$\mathbf{n}_1 = \langle 1,0,1\rangle \times \langle 1,1,0\rangle = \langle -1,1,1\rangle\,.$$

Similarly, we define the plane $\mathcal{P}_2$ to contain $(0,0,0)$, $(1,1,0)$ and $(0,1,1)$; a normal to this plane is

$$\mathbf{n}_2 = \langle 1,1,0\rangle \times \langle 0,1,1\rangle = \langle 1,-1,1\rangle\,.$$

The dihedral angle between these two planes is the smaller angle θ between $\mathbf{n}_1$ and $\mathbf{n}_2$, with

$$\cos\theta = \left|\frac{\mathbf{n}_1\cdot\mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|}\right| = \frac{1}{3}.$$

(We use the absolute value to ensure that we get the cosine of the smaller angle.) We conclude that

$$\theta = \cos^{-1}(1/3) \approx 1.23 \text{ or } \approx 70.5\,\text{deg}\,.$$

(Any other tetrahedron will be similar (in the geometric sense) to this one so it will have the same dihedral angle.)

2. (a) Calculating directly, we see

$$|\mathbf{u}+\mathbf{v}|^2 = (\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v}) \quad = \quad \mathbf{u}\cdot\mathbf{u} + 2\mathbf{u}\cdot\mathbf{v} + \mathbf{v}\cdot\mathbf{v},$$

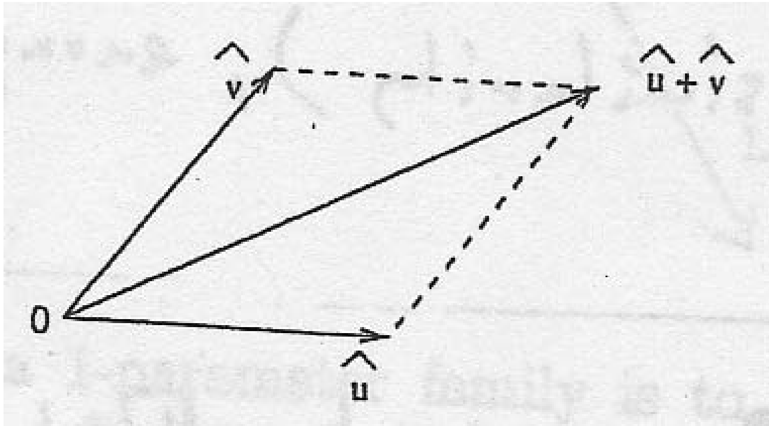
since $\mathbf{u}\cdot\mathbf{v} = \mathbf{v}\cdot\mathbf{u}$. Also,

$$|\mathbf{u}-\mathbf{v}|^2 = (\mathbf{u}-\mathbf{v})\cdot(\mathbf{u}-\mathbf{v}) \quad = \quad \mathbf{u}\cdot\mathbf{u} - 2\mathbf{u}\cdot\mathbf{v} + \mathbf{v}\cdot\mathbf{v}.$$

Subtracting these two equations gives

$$|\mathbf{u}+\mathbf{v}|^2 - |\mathbf{u}-\mathbf{v}|^2 = 4\mathbf{u}\cdot\mathbf{v}$$

as desired.



(b) Since $\mathbf{u}$ and $\mathbf{v}$ have the same length (=1), $\mathbf{u}+\mathbf{v}$ bisects the angle between $\mathbf{u}$ and $\mathbf{v}$. We make this a unit vector:

$$\frac{\mathbf{u}+\mathbf{v}}{|\mathbf{u}+\mathbf{v}|}.$$

See figure.

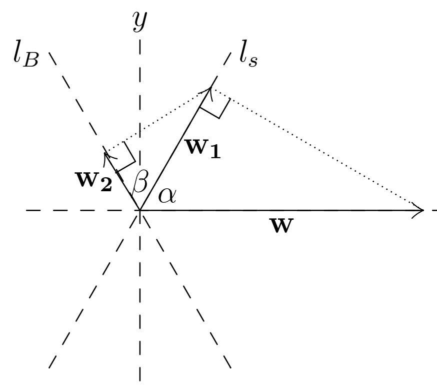
3. a) See the sketch with part (b). It is clear that no matter what the angle α the vector $\mathbf{w_1}$ has a rightward component.

b) In the sketch we've labeled the projection we want as $\mathbf{w_2}$.

We see easily that $|\mathbf{w_1}| = a\cos\alpha$ and $|\mathbf{w_2}| = \cos\beta|\mathbf{w_1}| = a\cos\beta\cos\alpha$.

Thus $\boxed{\mathbf{w_2} = a\cos\beta\cos\alpha\langle\cos(\alpha+\beta), \sin(\alpha+\beta)\rangle.}$

The component of $\mathbf{w_2}$ in the $\mathbf{i}$ direction is $a\cos(\alpha)\cos(\beta)\cos(\alpha+\beta)$. Since α and β are between 0 and $\pi/2$ this component is negative exactly when $\boxed{\alpha+\beta > \pi/2.}$ (This is easily seen in the sketch as well.)



18.02SC Multivariable Calculus
Fall 2010