Solutions for PSet 6

1. (8.22:14)

   (a) $f(x,y) = f_1(x,y)\mathbf{i} + f_2(x,y)\mathbf{j}$, where $f_1(x,y) = e^{x+2y}$, and $f_2(x,y) = \sin(y+2x)$. Computing all the partial derivatives

$$\frac{\partial f_1}{\partial x} = e^{x+2y} \qquad \frac{\partial f_1}{\partial y} = 2e^{x+2y}$$
$$\frac{\partial f_2}{\partial x} = 2\cos(y+2x) \qquad \frac{\partial f_2}{\partial y} = \cos(y+2x)$$

   So the matrix for the total derivative is:

$$Df(x,y) = \begin{pmatrix} e^{x+2y} & 2e^{x+2y} \\ 2\cos(y+2x) & \cos(y+2x) \end{pmatrix}$$

   Similarly for $g(u,v,w) = g_1(u,v,w)\mathbf{i} + g_2(u,v,w)\mathbf{j}$, where $g_1(u,v,w) = u + 2v^2 + 3w^3$ and $g_2(u,v,w) = 2v - u^2$ we have:

$$\frac{\partial g_1}{\partial u} = 1 \qquad \frac{\partial g_1}{\partial v} = 4v \qquad \frac{\partial g_1}{\partial w} = 9w^2$$
$$\frac{\partial g_2}{\partial u} = -2u \qquad \frac{\partial g_2}{\partial v} = 2 \qquad \frac{\partial g_2}{\partial w} = 0$$

   And the total derivative is:

$$Dg(u,v,w) = \begin{pmatrix} 1 & 4v & 9w^2 \\ -2u & 2 & 0 \end{pmatrix}$$

   (b) The composition

$$h(u,v,w) = f(g(u,v,w)) = exp(u+2v^2+3w^3+4v-2u^2)\mathbf{i}+\sin(2v-u^2+2u+4v^2+6w^3)\mathbf{j}$$

   (c) The total derivative at a point (u,v,w) can be computed using the chain rule:

$$\begin{aligned} Dh(u,v,w) &= Df(g(u,v,w))Dg(u,v,w) \\ &= \begin{pmatrix} e^{g_1+2g_2} & 2e^{g_1+2g_2} \\ 2\cos(g_2+2g_1) & \cos(g_2+2g_1) \end{pmatrix} \begin{pmatrix} 1 & 4v & 9w^2 \\ -2u & 2 & 0 \end{pmatrix} \end{aligned}$$

   Now we evaluate at $(u,v,w) = (1,-1,1)$ and thus $g_1 = 6, g_2 = -3$. As a result $g_1 + 2g_2 = 0$ and $g_2 + 2g_1 = 9$ and

$$Dh(1,-1,1) = \begin{pmatrix} e^0 & 2e^0 \\ 2\cos 9 & \cos 9 \end{pmatrix} \begin{pmatrix} 1 & -4 & 9 \\ -2 & 2 & 0 \end{pmatrix} =$$

$$\begin{pmatrix} -3 & 0 & 9 \\ 0 & -6\cos 9 & 18\cos 9 \end{pmatrix}$$

2. (8.24:12)

   (a) We can compute $\bigtriangledown(\frac{1}{r})$ using the chain rule for the functions $r : \mathbb{R}^3 \to \mathbb{R}$ defined by $r(\mathbf{r}) = \sqrt{\mathbf{r}\cdot\mathbf{r}}$ and $g : \mathbb{R} \to \mathbb{R}$ defined by $g(t) = \frac{1}{t}$. With these functions $\frac{1}{r} = g \circ r$ thus

$$\mathbf{A} \bigtriangledown (\frac{1}{r}) = \mathbf{A} \cdot g'(r) \bigtriangledown (\sqrt{\mathbf{r}\cdot\mathbf{r}}) = \mathbf{A} \cdot \frac{-1}{(r^2)} \cdot (\frac{2\mathbf{r}}{2\sqrt{\mathbf{r}\cdot\mathbf{r}}}) = -\frac{\mathbf{A}}{r^3} \cdot \mathbf{r}$$

   (b) To evaluate the left hand side in question, we need to first evaluate $\bigtriangledown\big(\mathbf{A} \bigtriangledown (\frac{1}{r})\big)$. Using part (a), this is equivalent to

$$\bigtriangledown\left(\frac{-\mathbf{A}\cdot\mathbf{r}}{r^3}\right) = \bigtriangledown\left(\frac{f(\mathbf{r})}{h(\mathbf{r})}\right)$$

   where $f(\mathbf{r}) = -\mathbf{A}\cdot\mathbf{r}$ and $h(\mathbf{r}) = r^3$. Both are real-valued functions, thus we can apply the rule for their fractions:

$$\bigtriangledown\left(-\frac{\mathbf{A}\cdot\mathbf{r}}{r^3}\right) = \bigtriangledown\left(\frac{f(\mathbf{r})}{h(\mathbf{r})}\right) = \frac{\bigtriangledown(f(\mathbf{r}))h(\mathrm{r}) - f(\mathbf{r}) \bigtriangledown (h(\mathbf{r}))}{(h(\mathbf{r}))^2}$$

   But

$$\bigtriangledown h(\mathbf{r}) = \bigtriangledown r^3 = \bigtriangledown(\mathbf{r}\cdot\mathbf{r})^{\frac{3}{2}} = \frac{3}{2}(\mathbf{r}\cdot\mathbf{r})^{\frac{1}{2}}2\mathbf{r} = 3r\mathbf{r}$$

   Therefore

$$\bigtriangledown\left(-\frac{\mathbf{A}\cdot\mathbf{r}}{r^3}\right) = \frac{\mathbf{A}r^3 - \mathbf{A}\cdot\mathbf{r}3r\mathbf{r}}{r^6}$$

   Now

$$\mathbf{B}\cdot\bigtriangledown\left(-\frac{\mathbf{A}\cdot\mathbf{r}}{r^3}\right) = \frac{3\mathbf{B}\cdot\mathbf{r}\mathbf{A}\cdot\mathbf{r}}{r^5} - \frac{\mathbf{A}\cdot\mathbf{B}}{r^3}$$

3. To compute the gradient of multivariate function $f(x, y)$, compute the partial derivatives:

$$\begin{aligned}\frac{\partial f(x,y)}{\partial x} &= \lim_{h\to 0}\frac{\int_0^{(x+h)y} g(u)du - \int_0^{xy} g(u)du}{h}\\ &= \lim_{h\to 0}\frac{\int_{xy}^{(x+h)y} g(u)du}{h}\end{aligned}$$

If we define a function $m(h) = \int_{xy}^{xy+hy} g(u)du$ then the limit above is $m'(h)$. Using the fundamental theorem of calculus, we determine $m'(0) = yg(xy)$. The partial derivative with respect to y follows similarly. Thus the gradient of $f(x, y) = \displaystyle\int_0^{xy} g(u)du$ is $\nabla f(x, y) = (yg(xy),\, xg(xy))$.

A level set (x, y) can be described as $f^{-1}(c)$. If both (x_0, y_0) and (x, y) lie in the same level set, then:

$$\begin{aligned} \int_0^{xy} g(u)du &= \int_0^{x_0y_0} g(u)du = c \\ \text{Or,} \quad \int_{x_0y_0}^{xy} g(u)du &= 0 \end{aligned}$$

As g is a positive function, its integral can only be 0 if the integration interval is empty or:

$$xy = x_0y_0 \quad \text{or} \quad \exists \text{ a value } b \text{ s.t. } xy = x_0y_0 = b$$

As g is positive, the function $m(t) = \displaystyle\int_0^t g(u)du$ is strictly increasing in t. Therefore, there exists a unique b such that $\displaystyle\int_0^b g(u)du = c \neq 0$. In other words, the level set is parametrized by $y = h(x) = \dfrac{b}{x}$ where b is unique.

A level set (x, y) can be parametrized as $\mathbf{r}(x) = f^{-1}(c) = (x, \dfrac{b}{x})$. This level set has slope $\mathbf{r}'(x)$ given by $(1, -\dfrac{b}{x^2})$. The gradient of $f(x, y)$, ∇f at any point (x, y) is $g(b)(\dfrac{b}{x}, x)$. The dot product $\nabla f \cdot \mathbf{r}'(x)$ is:

$$g(b)(\frac{b}{x}, x) \cdot (1, -\frac{b}{x^2}) = g(b)\frac{b}{x} - g(b)\frac{bx}{x^2} = 0$$

Hence, ∇f is orthogonal to the level set at each point on the curve.

4.

$$f(x, y) = \begin{cases} \frac{x^2-y^2}{x^2+y^2}xy & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$

The partial derivatives:

$$\frac{\partial f}{\partial x}(0,y) = \lim_{h\to 0}\frac{f(h,y)-f(0,y)}{h} = \lim_{h\to 0}\frac{h^2-y^2}{h^2+y^2}y = -y$$

$$\frac{\partial f}{\partial y}(x,0) = \lim_{h\to 0}\frac{f(x,h)-f(x,0)}{h} = \lim_{h\to 0}\frac{x^2-h^2}{x^2+h^2}x = x$$

Using the above derivation, the second partial derivatives can be evaluated at point (0,0):

$$\frac{\partial^2 f}{\partial x\partial y}(0,0) = \frac{\partial}{\partial x}\frac{\partial f}{\partial y}(x,0)\bigg|_{(0,0)} = \frac{\partial}{\partial x}(x)\bigg|_{(0,0)} = 1$$

and

$$\frac{\partial^2 f}{\partial y\partial x}(0,0) = \frac{\partial}{\partial y}\frac{\partial f}{\partial x}(0,y)\bigg|_{(0,0)} = \frac{\partial}{\partial y}(-y)\bigg|_{(0,0)} = -1$$

Thus

$$\frac{\partial^2 f}{\partial x\partial y}(0,0) \neq \frac{\partial^2 f}{\partial y\partial x}(0,0)$$

This means that in general,

$$\frac{\partial^2 f}{\partial x\partial y} \neq \frac{\partial^2 f}{\partial x\partial y}$$

5. We can write $F(t) = f(\mathbf{r}(t))$ where $\mathbf{r}(t) = (3t^2, 2t+1, 3-t^3)$.

   Then $F(t) = f\circ\mathbf{r}$, thus

$$F'(t) = \bigtriangledown f(3t^2, 2t+1, 3-t^3)\cdot\mathbf{r}'(t) = \bigtriangledown f(3t^2, 2t+1, 3-t^3)\cdot(6t, 2, -3t^2)$$

   At $t = 1$ this evaluates to:

$$F'(1) = \bigtriangledown f(3,3,2)\cdot(6,2,-3)$$

   The gradient of $f:\mathbb{R}^3\to\mathbb{R}$ is $\bigtriangledown f(x,y,z) = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z})$. Thus

$$F'(1) = 6\frac{\partial f}{\partial x}(3,3,2) + 2\frac{\partial f}{\partial y}(3,3,2) - 3\frac{\partial f}{\partial z}(3,3,2)$$

Let $Hess_f$ denote the second derivative matrix of f:

$$Hess_f(x,y,z) = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2}(x,y,z) & \frac{\partial^2 f}{\partial x \partial y}(x,y,z) & \frac{\partial^2 f}{\partial x \partial z}(x,y,z) \\ \frac{\partial^2 f}{\partial y \partial x}(x,y,z) & \frac{\partial^2 f}{\partial y^2}(x,y,z) & \frac{\partial^2 f}{\partial y \partial z}(x,y,z) \\ \frac{\partial^2 f}{\partial z \partial x}(x,y,z) & \frac{\partial^2 f}{\partial z \partial y}(x,y,z) & \frac{\partial^2 f}{\partial z^2}(x,y,z) \end{pmatrix}$$

Then

$$\begin{aligned} F''(t) &= \mathbf{r}'(t) Hess_f(3t^2, 2t+1, 3-t^3)\mathbf{r}'(t)^T + \bigtriangledown f(3t^2, 2t+1, 3-t^3) \cdot \mathbf{r}''(t) \\ &= (6t, 2, -3t^2) f''(3t^2, 2t+1, 3-t^3)(6t, 2, -3t^2)^T + \bigtriangledown f(3t^2, 2t+1, 3-t^3) \cdot (6, 0, -6t) \\ &= 36t^2 \frac{\partial^2 f}{\partial x^2} + 12t \frac{\partial^2 f}{\partial x \partial y} - 18t^3 \frac{\partial^2 f}{\partial x \partial z} + 12t \frac{\partial^2 f}{\partial y \partial x} + 4 \frac{\partial^2 f}{\partial y^2} - 6t^2 \frac{\partial^2 f}{\partial y \partial z} - 18t^3 \frac{\partial^2 f}{\partial z \partial x} \\ &\quad -6t^2 \frac{\partial^2 f}{\partial z \partial y} + 9t^4 \frac{\partial^2 f}{\partial z^2} + 6 \frac{\partial f}{\partial x} - 6t \frac{\partial f}{\partial z} \end{aligned}$$

where all partial derivatives are taken at $(3t^2, 2t+1, 3-t^3)$.

Substituting $t = 1$ we get

$$\begin{aligned} F''(t) &= 36 \frac{\partial^2 f}{\partial x^2} + 12 \frac{\partial^2 f}{\partial x \partial y} - 18 \frac{\partial^2 f}{\partial x \partial z} + 12 \frac{\partial^2 f}{\partial y \partial x} + 4 \frac{\partial^2 f}{\partial y^2} - 6 \frac{\partial^2 f}{\partial y \partial z} - 18 \frac{\partial^2 f}{\partial z \partial x} \\ &\quad -6 \frac{\partial^2 f}{\partial z \partial y} + 9 \frac{\partial^2 f}{\partial z^2} + 6 \frac{\partial f}{\partial x} - 6 \frac{\partial f}{\partial z} \end{aligned}$$

where the partial derivatives are taken at $(3,3,2)$.

6. (a) $h(\mathbf{x}) = g(f(\mathbf{x}))$. Thus $Dh((0,0)) =$

$$Dg(f((0,0)))Df(0,0) = Dg(1,2)Df(0,0) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} -1 & 2 \\ 6 & 3 \end{pmatrix} = \begin{pmatrix} 5 & 5 \\ 6 & 3 \\ 4 & 7 \end{pmatrix}$$

(b) Let $k = f^{-1}$, then $Dk(0,0) =$

$$\left(Df(f^{-1}(0,0))\right)^{-1} = (Df(1,2))^{-1}$$

Thus,

$$Dk(0,0) = \begin{pmatrix} -1 & 3 \\ -2 & 4 \end{pmatrix}^{-1} = \frac{1}{2} \begin{pmatrix} 4 & -3 \\ 2 & -1 \end{pmatrix}$$