1. Vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$

Definition 1.1. *A vector $\vec{v} \in \mathbb{R}^3$ is a 3-tuple of real numbers (v_1, v_2, v_3).*

Hopefully the reader can well imagine the definition of a vector in $\mathbb{R}^2$.

Example 1.2. $(1,1,0)$ *and* $(\sqrt{2}, \pi, 1/e)$ *are vectors in* $\mathbb{R}^3$.

Definition 1.3. *The* ***zero vector*** *in $\mathbb{R}^3$, denoted $\vec{0}$, is the vector $(0,0,0)$. If $\vec{v} = (v_1, v_2, v_3)$ and $\vec{w} = (w_1, w_2, w_3)$ are two vectors in $\mathbb{R}^3$, the* ***sum of*** *$\vec{v}$* ***and*** *$\vec{w}$, denoted $\vec{v} + \vec{w}$, is the vector $(v_1 + w_1, v_2 + w_2, v_3 + w_3)$.*

If $\vec{v} = (v_1, v_2, v_3) \in \mathbb{R}^3$ is a vector and $\lambda \in \mathbb{R}$ is a ***scalar****, the* ***scalar product of*** *λ* ***and*** *v, denoted $\lambda \cdot \vec{v}$, is the vector $(\lambda v_1, \lambda v_2, \lambda v_3)$.*

Example 1.4. *If $\vec{v} = (2, -3, 1)$ and $\vec{w} = (1, -5, 3)$ then $\vec{v} + \vec{w} = (3, -8, 4)$. If $\lambda = -3$ then $\lambda \cdot \vec{v} = (-6, 9, -3)$.*

Lemma 1.5. *If λ and μ are scalars and $\vec{u}$, $\vec{v}$ and $\vec{w}$ are vectors in $\mathbb{R}^3$, then*

(1) $\vec{0} + \vec{v} = \vec{0}$.
(2) $\vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}$.
(3) $\vec{u} + \vec{v} = \vec{v} + \vec{u}$.
(4) $\lambda \cdot (\mu \cdot \vec{v}) = (\lambda\mu) \cdot \vec{v}$.
(5) $(\lambda + \mu) \cdot \vec{v} = \lambda \cdot \vec{v} + \mu \cdot \vec{v}$.
(6) $\lambda \cdot (\vec{u} + \vec{v}) = \lambda \cdot \vec{u} + \lambda \cdot \vec{v}$.

Proof. We check (3). If $\vec{u} = (u_1, u_2, u_3)$ and $\vec{v} = (v_1, v_2, v_3)$, then

$$\begin{aligned} \vec{u} + \vec{v} &= (u_1 + v_1, u_2 + v_2, u_3 + v_3) \\ &= (v_1 + u_1, v_2 + u_2, v_3 + u_3) \\ &= \vec{v} + \vec{u}. \end{aligned}$$ □

We can interpret vector addition and scalar multiplication geometrically. We can think of a vector as representing a displacement from the origin. Geometrically a vector $\vec{v}$ has a *magnitude* (or length) $|\vec{v}| = (v_1^2 + v_2^2 + v_3^2)^{1/2}$ and every non-zero vector has a *direction*

$$\vec{u} = \frac{\vec{v}}{|\vec{v}|}.$$

Multiplying by a scalar leaves the direction unchanged and rescales the magnitude. To add two vectors $\vec{v}$ and $\vec{w}$, think of transporting the tail of $\vec{w}$ to the endpoint of $\vec{v}$. The sum of $\vec{v}$ and $\vec{w}$ is the vector whose tail is the tail of the transported vector.

One way to think of this is in terms of directed line segments. Note that given a point P and a vector $\vec{v}$ we can add $\vec{v}$ to P to get another point Q. If $P = (p_1, p_2, p_3)$ and $\vec{v} = (v_1, v_2, v_3)$ then

$$Q = P + \vec{v} = (p_1 + v_1, p_2 + v_2, p_3 + v_3).$$

If $Q = (q_1, q_2, q_3)$, then there is a unique vector $\overrightarrow{PQ}$, such that $Q = P + \vec{v}$, namely

$$\overrightarrow{PQ} = (q_1 - p_1, q_2 - p_2, q_3 - p_3).$$

Lemma 1.6. *Let P, Q and R be three points in $\mathbb{R}^3$.*
Then $\overrightarrow{PQ} + \overrightarrow{QR} = \overrightarrow{PR}$.

Proof. Let us consider the result of adding $\overrightarrow{PQ} + \overrightarrow{QR}$ to P,

$$\begin{aligned} P + (\overrightarrow{PQ} + \overrightarrow{QR}) &= (P + \overrightarrow{PQ}) + \overrightarrow{QR} \\ &= Q + \overrightarrow{QR} \\ &= R. \end{aligned}$$

On the other hand, there is at most one vector $\vec{v}$ such that when we add it P we get R, namely the vector $\overrightarrow{PR}$. So $\overrightarrow{PQ} + \overrightarrow{QR} = \overrightarrow{PR}$. $\square$

Note that (1.6) expresses the geometrically obvious statement that if one goes from P to Q and then from Q to R, this is the same as going from P to R.

Vectors arise quite naturally in nature. We can use vectors to represent forces; every force has both a magnitude and a direction. The combined effect of two forces is represented by the vector sum. Similarly we can use vectors to measure both velocity and acceleration. The equation

$$\vec{F} = m\vec{a},$$

is the vector form of Newton's famous equation.

Note that $\mathbb{R}^3$ comes with three standard unit vectors

$$\hat{\imath} = (1, 0, 0) \qquad \hat{\jmath} = (0, 1, 0) \qquad \text{and} \qquad \hat{k} = (0, 0, 1),$$

which are called the *standard basis*. Any vector can be written uniquely as a linear combination of these vectors,

$$\vec{v} = (v_1, v_2, v_3) = v_1\hat{\imath} + v_2\hat{\jmath} + v_3\hat{k}.$$

We can use vectors to parametrise lines in $\mathbb{R}^3$. Suppose we are given two different points P and Q of $\mathbb{R}^3$. Then there is a unique line l containing P and Q. Suppose that $R = (x, y, z)$ is a general point of the line. Note that the vector $\overrightarrow{PR}$ is parallel to the vector $\overrightarrow{PQ}$, so that $\overrightarrow{PR}$ is a scalar multiple of $\overrightarrow{PQ}$. Algebraically,

$$\overrightarrow{PR} = t\overrightarrow{PQ},$$

for some scalar $t \in \mathbb{R}$. If $P = (p_1, p_2, p_3)$ and $Q = (q_1, q_2, q_3)$, then

$$(x - p_1, y - p_2, z - p_3) = t(q_1 - p_1, q_2 - p_2, q_3 - p_3) = t(v_1, v_2, v_3),$$

where $(v_1, v_2, v_3) = (q_1 - p_1, q_2 - p_2, q_3 - p_3)$. We can always rewrite this as,

$$(x, y, z) = (p_1, p_2, p_3) + t(v_1, v_2, v_3) = (p_1 + tv_1, p_2 + tv_2, p_3 + tv_3).$$

Writing these equations out in coordinates, we get

$$x = p_1 + tv_1 \qquad y = p_2 + tv_2 \qquad \text{and} \qquad z = p_3 + tv_3.$$

Example 1.7. *If $P = (1, -2, 3)$ and $Q = (1, 0, -1)$, then $\vec{v} = (0, 2, -4)$ and a general point of the line containing P and Q is given parametrically by*

$$(x, y, z) = (1, -2, 3) + t(0, 2, -4) = (1, -2 + 2t, 3 - 4t).$$

Example 1.8. *Where do the two lines l_1 and l_2*

$$(x, y, z) = (1, -2 + 2t, 3 - 4t) \qquad \textit{and} \qquad (x, y, z) = (2t - 1, -3 + t, 3t),$$

intersect?

We are looking for a point (x, y, z) common to both lines. So we have

$$(1, -2 + 2s, 3 - 4s) = (2t - 1, -3 + t, 3t).$$

Looking at the first component, we must have $t = 1$. Looking at the second component, we must have $-2 + 2s = -2$, so that $s = 0$. By inspection, the third component comes out equal to 3 in both cases. So the lines intersect at the point $(1, -2, 3)$.

Example 1.9. *Where does the line*

$$(x, y, z) = (1 - t, 2 - 3t, 2t + 1)$$

intersect the plane

$$2x - 3y + z = 6?$$

We must have

$$2(1 - t) - 3(2 - 3t) + (2t + 1) = 6.$$

Solving for t we get

$$9t - 3 = 6,$$

so that $t = 1$. The line intersects the plane at the point

$$(x, y, z) = (0, -1, 3).$$

Example 1.10. *A cycloid is the path traced in the plane, by a point on the circumference of a circle as the circle rolls along the ground.*

Let's find the parametric form of a cycloid. Let's suppose that the circle has radius a, the circle rolls along the x-axis and the point is at the origin at time $t = 0$. We suppose that the cylinder rotates through an angle of t radians in time t. So the circumference travels a distance of at. It follows that the centre of the circle at time t is at the point $P = (at, a)$. Call the point on the circumference $Q = (x, y)$ and let O be the centre of coordinates. We have

$$(x, y) = \overrightarrow{OQ} = \overrightarrow{OP} + \overrightarrow{PQ}.$$

Now relative to P, the point Q just goes around a circle of radius a. Note that the circle rotates backwards and at time $t = 0$, the angle $3\pi/2$. So we have

$$\overrightarrow{PQ} = (a\cos(3\pi/2 - t), a\sin(3\pi/2 - t)) = (-a\sin t, -a\cos t)$$

Putting all of this together, we have

$$(x, y) = (at - a\sin t, a - a\cos t).$$

18.022 Calculus of Several Variables
Fall 2010