# 18.02 Practice Final 3 hrs.

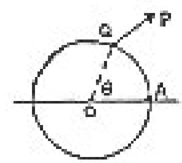
**Problem 1.** Given the points $P:(1,1,-1), Q:(1,2,0), R:(-2,2,2)$, find

a) $PQ \times PR$  b) a plane $ax+by+cz=d$ through $P, Q, R$.

**Problem 2.** Let $A = \begin{pmatrix} 1 & 0 & c \\ 2 & c & 1 \\ 1 & -1 & 2 \end{pmatrix}$, $\mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, $\mathbf{0} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, $A^{-1} = \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \times & \cdot \end{pmatrix}$.

a) For what value(s) of the constant $c$ will $A\mathbf{x} = \mathbf{0}$ have a non-zero solution?

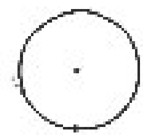
b) Take $c = 2$, and tell what entry the inverse matrix has in the position marked $\times$.




**Problem 3.** The roll of Scotch tape shown has outer radius $a$ and is fixed in position (i.e., does not turn). Its end $P$ is originally at the point $A$; the tape is then pulled from the roll so the free portion makes a 45-degree angle with the horizontal.

Write parametric equations $x = x(\theta), y = y(\theta)$ for the curve $C$ traced out by the point $P$ as it moves. (Use vector methods; $\theta$ is the angle shown).

Sketch the curve on the second picture, showing its behavior at its endpoints.



**Problem 4.** The position vector of a point $P$ is $\mathbf{r} = \langle 3\cos t, 5\sin t, 4\cos t \rangle$.

a) Show its speed is constant.

b) At what point $A:(a,b,c)$ does $P$ pass through the $yz$-plane?

**Problem 5.** Let $w = x^2y - xy^2$, and $P = (2,1)$.

a) Find the directional derivative $\dfrac{dw}{ds}$ at $P$ in the direction of $\mathbf{A} = 3\,\mathbf{i} + 4\,\mathbf{j}$.

b) If you start at $P$ and go a distance .01 in the direction of $\mathbf{A}$, by approximately how much will $w$ change? (Give a decimal with one significant digit.)

**Problem 6.** a) Find the tangent plane at $(1,1,1)$ to the surface $x^2 + 2y^2 + 2z^2 = 5$; give the equation in the form $ax + by + cz = d$ and simplify the coefficients.

b) What dihedral angle does the tangent plane make with the $xy$-plane? (Hint: consider the normal vectors of the two planes.)

**Problem 7.** Find the point on the plane $2x + y - z = 6$ which is closest to the origin, by using Lagrange multipliers. (Minimize the square of the distance. Only 10 points if you use some other method.)

**Problem 8.** Let $w = f(x,y,z)$ with the constraint $g(x,y,z) = 3$.

At the point $P:(0,0,0)$, we have $\nabla f = \langle 1,1,2 \rangle$ and $\nabla g = \langle 2,-1,-1 \rangle$. Find the value at $P$ of the two quantities (show work): a) $\left(\dfrac{\partial z}{\partial x}\right)_y$ b) $\left(\dfrac{\partial w}{\partial x}\right)_y$

**Problem 9.** Evaluate by changing the order of integration: $\displaystyle\int_0^3 \int_{x^2}^9 x e^{-y^2}\, dy\, dx$.

**Problem 10.** A plane region $R$ is bounded by four semicircles of radius 1, having ends at $(1,1),(1,-1),(-1,1),(-1,-1)$ and centerpoints at $(2,0),(-2,0),(0,2),(0,-2)$.

Set up an iterated integral in polar coordinates for the moment of inertia of $R$ about the origin; take the density $\delta = 1$. Supply integrand and limits, but *do not evaluate* the integral. Use symmetry to simplify the limits of integration.

**Problem 11.** a) In the $xy$-plane, let $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j}$. Give in terms of $P$ and $Q$ the line integral representing the flux of $\mathbf{F}$ across a simple closed curve $C$, with outward-pointing normal.

b) Let $\mathbf{F} = ax\,\mathbf{i} + by\,\mathbf{j}$. How should the constants $a$ and $b$ be related if the flux of $\mathbf{F}$ over any simple closed curve $C$ is equal to the area inside $C$?

**Problem 12.** A solid hemisphere of radius 1 has its lower flat base on the $xy$-plane and center at the origin. Its density function is $\delta = z$. Find the force of gravitational attraction it exerts on a unit point mass at the origin.

**Problem 13.** Evaluate $\int_C (y-x)dx + (y-z)dz$ over the line segment $C$ from $P:(1,1,1)$ to $Q:(2,4,8)$.

**Problem 14.** a) Let $\mathbf{F} = ay^2\,\mathbf{i} + 2y(x+z)\,\mathbf{j} + (by^2+z^2)\,\mathbf{k}$. For what values of the constants $a$ and $b$ will $\mathbf{F}$ be conservative? Show work.

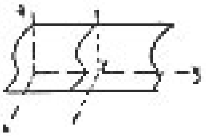
b) Using these values, find a function $f(x,y,z)$ such that $\mathbf{F} = \nabla f$.

c) Using these values, give the equation of a surface $S$ having the property: $\int_P^Q \mathbf{F}\cdot d\mathbf{r} = 0$ for any two points $P$ and $Q$ on the surface $S$.

**Problem 15.** Let $S$ be the closed surface whose bottom face $B$ is the unit disc in the $xy$-plane and whose upper surface is the paraboloid $z = 1 - x^2 - y^2$, $z \geq 0$. Find the flux of $\mathbf{F} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$ across $U$ by using the divergence theorem.

**Problem 16.** Using the data of the preceding problem, calculate the flux of $\mathbf{F}$ across $U$ directly, by setting up the surface integral for the flux and evaluating the resulting double integral in the $xy$-plane.

**Problem 17.** An $xz$-cylinder in 3-space is a surface given by an equation $f(x,z) = 0$ in $x$ and $z$ alone; its section by any plane $y = c$ perpendicular to the $y$-axis is always the same $xz$-curve. (See picture.)



Show that if $\mathbf{F} = x^2\,\mathbf{i} + y^2\,\mathbf{j} + xz\,\mathbf{k}$, then $\oint_C \mathbf{F}\cdot d\mathbf{r} = 0$ for any simple closed curve $C$ lying on an $xz$-cylinder. (Use Stokes' theorem.)

**Problem 18.** $\int e^{-x^2}dx$ is not elementary but $I = \int_0^\infty e^{-x^2}dx$ can still be evaluated.

a) Evaluate the iterated integral $\int_0^\infty \int_0^\infty e^{-x^2}e^{-y^2}\,dy\,dx$, in terms of $I$.

b) Then evaluate the integral in (a) by switching to polar coordinates. Comparing the two evaluations, what value do you get for $I$?