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18.02 Multivariable Calculus
Fall 2007

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18.02 Problem Set 11

Due Thursday 11/29/07, 12:45 pm.

Part A (17 points)

Hand in the underlined problems only; the others are for more practice.

Lecture 30. Thu Nov. 15 Vector fields in 3D; surface integrals and flux.

Read: Notes V8, V9.

Work: 6A/1, 2, 3, 4; 6B/1, 2, 3, 4, 6, 8.

Lecture 31. Fri Nov. 16 Divergence theorem.

Read: Notes V10; 15.6.

Work: 6C/1a, 2, 3, 5, 6, 7a, 8.

Lecture 32. Tue Nov. 20 Divergence theorem continued: applications and proof.

Read: Notes V10; 15.6, and pp. 1054–1055 about heat flow.

Lecture 33. Tue Nov. 27 Line integrals in space, curl, exactness and potentials.

Read: Notes V11, V12; p. 1017–1018 on curl.

Work: 6D/1abcd, 2, 4, 5; 6E/1, 2, 3ab(ii) (both methods), 5.

Part B (26 points)

Directions: Attempt to solve *each part* of each problem yourself. If you collaborate, solutions must be written up independently. It is illegal to consult materials from previous semesters. With each problem is the day it can be done.

Write the names of all the people you consulted or with whom you collaborated and the resources you used.

Problem 1. (Thursday 11/15, 2 points) Notes 6B/7.

Problem 2. (Friday 11/16, 8 points: 1+2+3+2)

Consider a tetrahedron with vertices at $P_0 = (0, 0, 0)$, $P_1 = (1, 0, 1)$, $P_2 = (1, 0, -1)$, and $P_3 = (1, 1, 0)$.

a) Which two faces are exchanged by the symmetry $z \rightarrow -z$?

b) Find normals to each face (use the easiest ones, not unit normals), pointing outwards. (You can avoid one calculation using symmetry.)

c) Calculate the flux of $\vec{F} = y\hat{i}$ through each face. (Because \vec{F} is symmetric with respect to $z \rightarrow -z$, you can again avoid one calculation using symmetry.)

d) Verify the divergence theorem for the tetrahedron and the vector field \vec{F} by computing each side of the formula.

Problem 3. (Friday 11/16, 5 points: 2+1+2)

a) Let $f(x, y, z) = 1/\rho = (x^2 + y^2 + z^2)^{-1/2}$. Calculate $\vec{F} = \nabla f$, and describe geometrically the vector field \vec{F} .

b) Evaluate the flux of \vec{F} over the sphere of radius a centered at the origin.

c) Show that $\text{div } \vec{F} = 0$. Does the answer obtained in (b) contradict the divergence theorem? Explain.

Problem 4. (Tuesday 11/20, 6 points: 1+2+2+1)

The *Laplacian* of a function f is the quantity $\nabla^2 f = f_{xx} + f_{yy} + f_{zz}$; a function f is *harmonic* if it satisfies the Laplace equation $\nabla^2 f = 0$.

a) Show that, if S is a closed surface bounding a region D , and if f has continuous second derivatives inside D , then $\iint_S \nabla f \cdot \hat{\mathbf{n}} \, dS = \iiint_D \nabla^2 f \, dV$. (In particular, the flux of the gradient of a harmonic function through a closed surface is always zero).

b) Show that, if S is a closed surface bounding a region D , and if f has continuous second derivatives inside D , then $\iint_S f \nabla f \cdot \hat{\mathbf{n}} \, dS = \iiint_D (f \nabla^2 f + |\nabla f|^2) \, dV$.

c) Use the result of (b) to show that, if a function f is harmonic everywhere in D , and f is zero at every point of the boundary S , then f is zero everywhere in D . (first show that $\nabla f = 0$).

d) Deduce that if two functions f and g are harmonic everywhere in D , and $f = g$ on the boundary S , then $f = g$ everywhere in D .

Problem 5. (Tuesday 11/27, 5 points: 2+1+2)

a) Compute (in terms of the constants a, b) the work done by the vector field $\vec{F} = (a \sin z + bxy^2) \hat{\mathbf{i}} + 2x^2y \hat{\mathbf{j}} + (x \cos z - z^2) \hat{\mathbf{k}}$ along the portion of helix $x = \cos t$, $y = \sin t$, $z = t$ from $(1, 0, 0)$ to $(1, 0, 2\pi)$.

b) Compute $\text{curl } \vec{F}$. For which value(s) of a and b is the vector field \vec{F} conservative?

c) Let a and b be the values you found above. Find a potential function for \vec{F} using a systematic method, and verify the answer you found in part (a) using the fundamental theorem of calculus.