Stacking Blocks

Is it possible to stack blocks as shown in Figure 1, so that no part of the bottom block is below the top block? In general, how much horizontal distance can there be between the top block and the bottom block?

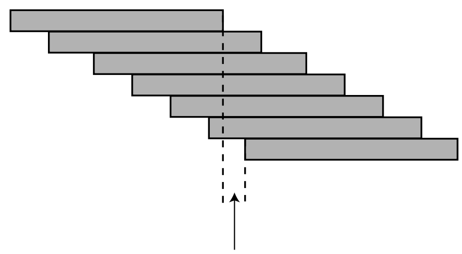

Figure 1: Stack of blocks.

This is a good kind of math question to ask. If there's a limit to how distant the top block can be from the bottom block it will be interesting to know what it is. It would also be interesting to discover that there's no limit to this distance. In the end, this becomes a question of whether the top block's position converges or diverges.

Professor Jerison has eight blocks; do you think he can achieve his goal with what he has?

In order to get the greatest possible horizontal distance, start at the top of the stack and work downward. The topmost block has to cover at least half of the block below it or else it will fall. Slide it so that its right end is at the midpoint of the block below it.

Next, slide the second block down as far to the left as you can without upsetting the tower. Then slide the block below it to the left as far as you can, and the one below that, and the one below that, and so on. In this way, Professor Jerison accomplished his goal using only 7 blocks.

How much further could we get if we had more blocks? Let's calculate it.

Calculation

To make the calculations simple, let's say that each block is 2 units long. Then if the left end of the topmost block is at position 0, then the left end of the block under it is at position 1.

In general the center of mass of the top n blocks must always be above the block supporting them.

- Let C_1 = the x coordinate of the center of mass of the top block.

- Let C_2 = the x coordinate of the center of mass of the top two blocks.
- Put the left end of the next block below the center of mass of the previous ones.

Question: How do you know that this is the best way to stack them?

Answer: I can't answer that in general, but I can tell you that this is the best we can do if we start building from the top – we're using what computer scientists call "the greedy algorithm" and going as far as we can at each step.

If we tried using this algorithm starting from the bottom it wouldn't work. We'd stack the second block with its end on the midpoint of the first block and then be unable to gain any distance beyond that.

It seems possible that there is some other strategy that's better than using the greedy algorithm starting at the top. There isn't, but we're not going to prove that today.

According to our strategy we need to know C_N, the x coordinate of the center of mass of the top N blocks, in order to continue.

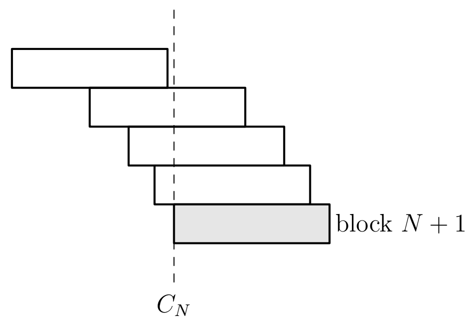

Figure 2: Adding a block.

If the center of mass of the top N blocks is on the line $x = C_N$, the center of mass of the $(N+1)^{st}$ block will have x coordinate $C_N + 1$. This shifts the center of mass of the stack to the right; the x coordinate of the new center of mass of the top $N+1$ blocks is given by the weighted average of the centers of mass of the stack:

$$\begin{aligned} C_{N+1} &= \frac{NC_N + 1(C_N+1)}{N+1} \\ &= \frac{(N+1)C_N + 1}{N+1} \\ C_{N+1} &= C_N + \frac{1}{N+1}. \end{aligned}$$

Adding the $(N+1)^{st}$ block added to the stack allows you to extend the stack $\frac{1}{N+1}$ units farther from its base.

So

$$\begin{aligned}
C_1 &= 1 \\
C_2 &= 1+\frac{1}{2} \\
C_3 &= 1+\frac{1}{2}+\frac{1}{3} \\
C_4 &= 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4} \\
C_5 &= 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5} > 2.
\end{aligned}$$

It takes at least 5 blocks to extend the top block beyond the base.

$$C_N = 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\cdots+\frac{1}{N}$$

This sum C_N is the same as S_N from a previous lecture:

$$C_N = S_N = \sum_{n=1}^{N}\frac{1}{n}.$$

We know that:

$$\ln N < S_N < (\ln N) + 1.$$

Since $\ln N$ goes to infinity as N goes to infinity, $S_N = C_N$ must go to infinity as N does. If we have enough blocks we *can* extend our stack as far as we want.

In this example, the fact that $\sum_{n=1}^{N}\frac{1}{n}$ diverges means that it's possible to extend the stack as far to the left as we wish, provided we have enough blocks.

On the other hand, the inequality $S_N < (\ln N) + 1$ tells us that it will take a lot of blocks to extend the top of the stack very far.

How high would this stack of blocks be if it extended across the two lab tables at the front of the lecture hall? One lab table is 6.5 blocks, or 13 units, long. Two tables are 26 units long. There will be $26 - 2 = 24$ units of overhang in the stack. (We subtract 2 because the bottom block has no overhang and because the stack extends one unit past the center of mass of the top block.) Each block is approximately 3 centimeters tall.

If $\ln n = 24$ then $n = e^{24}$ and:

$$\text{Height} = 3cm \cdot e^{24} \approx 8 \times 10^8 m.$$

That height is roughly twice the distance to the moon.

If you want the stack to span this room ($\sim$ 30 ft.) it would have to be 10^{26} meters high. That's about the diameter of the observable universe.

We can learn one more thing from this experiment — if we look at the stack sideways we see that it follows the shape of the graph of $\ln x$. This experiment provides a concrete example of how slowly the function $\ln x$ increases.

We did not discover an important number that limited the reach of the stack, but we did discover that that reach is infinite — infinity is also an important number. We also discovered a property of that infinite value; that the rate of extension of the stack is very slow. Infinity doesn't have a single value; there are lots of different orders of infinity.