Examples of Comparison

Example: $\displaystyle\sum_{n=0}^{\infty} \frac{1}{\sqrt{n^2+1}}$

We know that $\frac{1}{\sqrt{n^2+1}}$ is comparable to $\frac{1}{\sqrt{n^2}} = \frac{1}{n}$, so by limit comparison we know that $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+1}}$ converges or diverges as $\sum_{n=0}^{\infty} \frac{1}{n}$ does. We proved earlier that $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, so $\sum_{n=0}^{\infty} \frac{1}{\sqrt{n^2+1}}$ must diverge as well.

Note that we can include the $n = 0$ term in $\sum_{n=0}^{\infty} \frac{1}{\sqrt{n^2+1}}$ but not in $\sum_{n=1}^{\infty} \frac{1}{n}$. This is ok; the limit comparison test is is concerned only with long term behavior, not with the early partial sums.

Example: $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\sqrt{n^5-n^2}}$

Because we have a subtraction in the denominator we have to be careful not to divide by zero; we start our series at $n = 2$.

We compare $\frac{1}{\sqrt{n^5-n^2}}$ to $\frac{1}{\sqrt{n^5}} = \frac{1}{n^{5/2}}$. If we choose the right function to be the numerator, it's relatively simple to show that they are similar:

$$
\begin{aligned}
\lim_{n\to\infty} \frac{f(n)}{g(n)} &= \lim_{n\to\infty} \frac{\frac{1}{\sqrt{n^5}}}{\frac{1}{\sqrt{n^5-n^2}}} \\
&= \lim_{n\to\infty} \frac{\sqrt{n^5-n^2}}{\sqrt{n^5}} \\
&= \lim_{n\to\infty} \sqrt{\frac{n^5}{n^5} - \frac{n^2}{n^5}} \\
&= \sqrt{1-0} \\
&= 1.
\end{aligned}
$$

Since the two functions are similar, we can apply the limit comparison test and conclude that because $\sum_{n=1}^{\infty} \frac{1}{n^{5/2}}$ converges ($\frac{5}{2} > 1$), $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n^5-n^2}}$ must also converge.