Divergent Series

As with indefinite integrals, we're concerned about when infinite series converge. We're also interested in what goes wrong when a series diverges — when it fails to converge. Recall that when $|a| < 1$,

$$1 + a + a^2 + a^3 + \cdots = \frac{1}{1-a}.$$

How does this fail when $|a| \geq 1$?

One simple example of a divergent series is a geometric series with a equal to 1:

$$1 + 1 + 1 + 1 + \cdots = \frac{1}{1-1} = \frac{1}{0}.$$

This almost makes sense! Since the sum is infinite we conclude that the series diverge.

Now let's try $a = -1$. We get:

$$1 - 1 + 1 - 1 + \cdots$$

If we look at the partial sums of this sequence we see that they alternate between 1 and 0. If we plug $a = -1$ into our formula $\frac{1}{1-a}$ it predicts that the sum is $\frac{1}{2}$ which is halfway between 0 and 1 but is still wrong. Using the formula here is the equivalent of computing an indefinite integral without checking for singularities; it gives you an interesting but wrong result.

Because the partial sums of the series alternate between 0 and 1 without ever tending toward a single number, we say that this series is also divergent. The geometric series only converges when $|a| < 1$.

We'll look at one more case: $a = 2$. According to the formula,

$$1 + 2 + 2^2 + 2^3 + \cdots = \frac{1}{1-2} = -1.$$

This is clearly wrong. The sequence diverges; the left hand side is obviously infinite and the right hand side is -1. But number theorists actually have a way of making sense out of this, if they're willing to give up on the idea that 0 is less than 1.