Non-Constant Speed Parametrization

Let's look at the following parametrization:

$$\begin{aligned} x &= 2\sin t \\ y &= \cos t. \end{aligned}$$

To solve this sort of problem we're going to need to convert this parametrization in terms of cosine, sine and t into a rectangular equation in x and y. (We'll see a few more examples of this process later today in a different context.)

To see the pattern here we'll use the relationship:

$$\sin^2 t + \cos^2 t = 1.$$

Since $\frac{x}{2} = \sin t$, we get that:

$$\frac{1}{4}x^2 + y^2 = \sin^2 t + \cos^2 t = 1.$$

So the trajectory described by this parametrization is an ellipse.

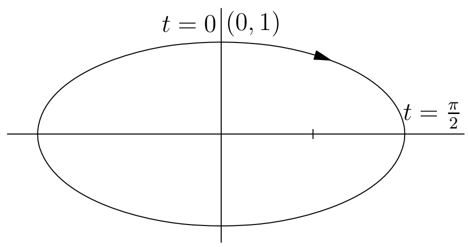

Figure 1: Ellipse described by $x = 2\sin t$, $y = \cos t$.

To sketch the ellipse we'll start by plotting a few points. When $t = 0$ we have:

$$(2\sin 0, \cos 0) = (0, 1),$$

so the ellipse "starts" at the point $(0, 1)$. When $t = \pi/2$ we get the point

$$\left(2\sin\frac{\pi}{2}, \cos\frac{\pi}{2}\right) = (2, 0).$$

We know that the trajectory follows an ellipse, and we can compute that the length of the minor axis is 1 and the major axis is 2, so we get the curve shown in Figure 1.

We also know that the motion is clockwise.

Next, let's examine the speed at which the point traces out arc length. We know:

$$\begin{aligned} \frac{ds}{dt} &= \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \\ &= \sqrt{(2\cos t)^2 + (-\sin t)^2} \end{aligned}$$

And so the total arc length covered by the point as it moves all the way around the ellipse (t varies from 0 to 2π) is:

$$\text{Arc length} = \int_0^{2\pi} \sqrt{4\cos^2 t + \sin^2 t}\, dt.$$

Unfortunately, this is not an elementary integral; we won't be able to find a formula for an antiderivative of $\sqrt{4\cos^2 t + \sin^2 t}$. That means we have to stop here; this is our final answer.

Note that it may be hard to tell whether or not it's possible to find a given antiderivative.

Question: When you draw the ellipse, don't you need to take into account what t is?

Answer: Good question. Our problem is to plot the curve parametrized by:

$$\begin{aligned} x &= 2\sin t \\ y &= \cos t. \end{aligned}$$

Obviously we can't simply graph y as a function of x; both y and x are functions of t. To draw this curve we did two things:

1. Plot points $(x(t), y(t))$ for "easy" values of t. (Here $t = 0$ and $t = \frac{\pi}{2}$.)

2. Find a relationship between x and y similar to one we can graph.

A calculator or computer would draw the curve by repeating the first step many, many times. We're not so patient, so we used the fact that $\left(\frac{1}{2}x\right)^2 + y^2 = 1$ to deduce that the curve must have an elliptical shape. Even if you don't recognize this as the equation of an ellipse, you should be able to guess that its graph will be a deformed circle.

By combining the information we have about locations of points on the curve (we might also want to find points for $t = \pi$ and $t = \frac{3\pi}{2}$) with the information that the curve looks like a deformed circle, we can get a fairly decent sketch of the parametrized curve.

You might be asked to give the rectangular equation for a parametric curve and then plot the curve. In this case, the answer would be $\frac{1}{4}x^2 + y^2 = 1$ followed by a picture of the ellipse.

Question: Do I need to know any specific formulas?

Answer: Any formulas that you know and remember may help you. You're not required to memorize the general equation of an ellipse, but you should recognize the equation of a circle.

Question: Arc length is the integral of $ds = \sqrt{dx^2 + dy^2}$. With ds, dy and dx all in the mix, how were you able to integrate just with respect to x in some examples?

Answer: When working with one dimensional objects like curves in space or in the plane, we're going to integrate with respect to a single variable. We get to choose which variable to use, but some choices are better than others.

For instance, if we're trying to find the arc length of a circle or the ellipse we just looked at, integrating with respect to x might be a problem because there are two different y values for most choices of x. However, if we can solve the problem by looking only at the top half of the circle or ellipse it could be ok to integrate with respect to x.

The uniform parameter (the one for which the point moves with uniform speed) may be the easiest one. In the previous example, the uniform parameter was t.

This returns us to the point that we're no longer tied to the view that y is a function of x. The variables x and y just represent the horizontal and vertical position of a point; they're no longer the input and output of a function. In fact, in this example both x and y are functions of t.