A Reduction Formula

When using a *reduction formula* to solve an integration problem, we apply some rule to rewrite the integral in terms of another integral which is a little bit simpler. We may have to rewrite that integral in terms of another integral, and so on for n steps, but we eventually reach an answer.

For example, to compute:

$$\int (\ln x)^n \, dx$$

we repeat the integration by parts from the previous example $n-1$ times, until we're just calculating $\int (\ln x) \, dx$.

For our first step we use:

$$\begin{aligned} u = (\ln x)^n \qquad & u' = n(\ln x)^{n-1}\frac{1}{x} \\ v = x \qquad & v' = 1. \end{aligned}$$

Then:

$$\begin{aligned} \int (\ln x)^n \, dx &= x(\ln x)^n - n\int (\ln x)^{n-1}\frac{1}{x}x \, dx \\ &= x(\ln x)^n - n\int (\ln x)^{n-1} \, dx \end{aligned}$$

So, if:

$$F_n(x) = \int (\ln x)^n \, dx$$

then we've just shown that:

$$F_n(x) = x(\ln x)^n - nF_{n-1}(x).$$

This is an example of a reduction formula; by applying the formula repeatedly we can write down what $F_n(x)$ is in terms of $F_1(x) = \int \ln x dx$ or $F_0(x) = \int 1 \, dx$.

We illustrate the use of a reduction formula by applying this one to the preceding two examples. We start by computing $F_0(x)$ and $F_1(x)$:

$$\begin{aligned} F_0(x) &= \int (\ln x)^0 \, dx = x + c \\ F_1(x) &= x(\ln x)^1 - 1F_0(x) \quad \text{(use reduction formula)} \\ &= x\ln x - x + c \quad \text{(Example 1)} \\ F_2(x) &= x(\ln x)^2 - 2F_1(x) \quad \text{(use reduction formula)} \\ &= x(\ln x)^2 - 2(x\ln x - x) + c \\ &= x(\ln x)^2 - 2x\ln x + 2x + c \quad \text{(Example 2.)} \end{aligned}$$

This is how reduction formulas work in general.