$$\int \frac{x-11}{(x^2+9)(x+2)}\,dx$$

Compute the integral:

$$\int \frac{x-11}{(x^2+9)(x+2)}\,dx.$$

Solution

The numerator and denominator of this integrand are fully factored and no terms cancel. The degree of the numerator is less than the degree of the denominator, so the integrand is in the correct form for the application of partial fractions decomposition.

One of the factors in the denominator is quadratic; the corresponding rational expression in our decomposition must have a linear numerator:

$$\frac{x-11}{(x^2+9)(x+2)} = \frac{Ax+B}{x^2+9} + \frac{C}{x+2}.$$

We can use the cover-up method to find C:

$$\frac{-2-11}{(-2)^2+9} = C \quad \Longrightarrow \quad C = -1.$$

Next we solve for A and B by multiplying both side of the decomposition equation by $(x^2+9)(x+2)$:

$$\begin{aligned}
\frac{x-11}{(x^2+9)(x+2)} &= \frac{Ax+B}{x^2+9} - \frac{1}{x+2} \\
x-11 &= (Ax+B)(x+2) - (x^2+9) \\
x^2+x-2 &= (Ax+B)(x+2) \\
(x-1)(x+2) &= (Ax+B)(x+2) \\
x-1 &= Ax+B.
\end{aligned}$$

We conclude that $A=1$ and $B=-1$, so:

$$\frac{x-11}{(x^2+9)(x+2)} = \frac{x-1}{x^2+9} - \frac{1}{x+2}.$$

We must now integrate this expression. In the course of this integration we will use the substitutions $u = x^2+9$ (so $x\,dx = du/2$), $x = 3\tan\theta$ (so $dx = 3\sec^2\theta\,d\theta$) and $v = x+2$.

$$\begin{aligned}
\int \frac{x-11}{(x^2+9)(x+2)}\,dx &= \int \frac{x-1}{x^2+9} - \frac{1}{x+2}\,dx \\
&= \int \frac{x}{x^2+9}\,dx - \int \frac{1}{x^2+9}\,dx - \int \frac{1}{x+2}\,dx
\end{aligned}$$

$$
\begin{aligned}
&= \int \frac{du/2}{u} - \int \frac{3\sec^2\theta}{9\tan^2\theta + 9}\, d\theta - \int \frac{1}{v}\, dv \\
&= \frac{1}{2}\ln|u| - \int \frac{3\sec^2\theta}{9\sec^2\theta}\, d\theta - \ln|v| \\
&= \frac{1}{2}\ln|u| - \int \frac{1}{3}\, d\theta - \ln|v| \\
&= \frac{1}{2}\ln|u| - \frac{1}{3}\,\theta - \ln|v| + c \\
&= \frac{1}{2}\ln|x^2+9| - \frac{1}{3}\, \tan^{-1}\left(\frac{x}{3}\right) - \ln|x+2| + c
\end{aligned}
$$

We see that even a relatively simple rational expression can have a complex integral!