Hi. Welcome back to recitation. I, today I wanted to teach you a variation on trig substitution. So this is called hyperbolic trigonometric substitution. And I'm going to teach you it just by going through a nice example of a question where it turns out to be useful.

So the question is the following. Compute the area of the region below. So what's the region? Well, I have here the hyperbola x squared minus y squared equals 1. And I've chosen a point on the hyperbola whose coordinates are (cosh t, sinh t). So remember that cosh t is a hyperbolic cosine, and sinh t is the hyperbolic sine. And they're given by the formulas cosh t equals e to the t plus e to the minus t over 2, and sinh t equals e to the t minus e to the minus t over 2. So and we saw in an earlier recitation video that this point, (cosh t, sinh t), is a point on the right branch of this hyperbola.

So I've got a region. So I've got the hyperbola, I've got that point. I've drawn a straight line between the origin and that point. So the region that I want you to find the area of is this region here. So it's the region below that line segment, above the x-axis, and to the left of this branch of the hyperbola.

Now, I'm not going to, I'm not going to just ask you to do that alone, because there's this technique that I want you to use. So let's start setting up the integral together, and then I'll describe the technique and give you a chance to work it out yourself, to work out the problem yourself.

So from looking at this region-- so let's think about computing the area of this region. There are two ways we could split it up, right? We could cut it up into vertical rectangles and integrate with respect to x, or we could cut it up into horizontal rectangles and integrate with respect to y. Now, if we cut it up into vertical rectangles, our life is a little complicated, because we have to cut the region into 2 pieces here. Right? There's the-- so this is the point (1, 0), where the hyperbola crosses x-axis. So over here, you know, the top part is the line segment and the bottom part is the x-axis, and then over here, the top part is the line segment and the bottom part is the hyperbola.

So life is complicated if we use vertical rectangles. It's a little bit simpler if we use horizontal rectangles, so let's go with that. You know, the amount of work will be similar either way, but I like this way, lets you right down in one integral.

So the area-- OK. So if we use horizontal rectangles to compute the area of a region, then the area is, we need to integrate from the bottom to the top, whatever the, you know, the bounds on y are. And then the thing we have to integrate has to be the area of one of those little rectangles. So the area of the little rectangle, its height is dy, and its length, or width, I guess, is the x-coordinate of its rightmost point, minus the x coordinate of its leftmost point.

So OK. So we need to integrate from the lowest value of y to the highest value. So we start at the bottom at y equals 0, and we need to go all the way up to the top here, which is y equals sinh t. So it's an integral from 0 to sinh t. OK. And so we need the x-coordinate on the right here, minus the x-coordinate on the left. So what's the x-coordinate on the right? Well, we need to solve this equation for x in terms of y. Right? This is going to be an integral with respect to y. So that if we solve for this point, we get x is equal to the square root of y squared plus 1. So the right coordinate is the square root of y squared plus 1. And the left coordinate, this is just a straight line passing through the origin, so its equation is y equals sinh t over cosh t times x, or x equals cosh t over sinh t times y. So this is minus cosh t over sinh t times y. And we're integrating with respect to y.

OK. So this is the integral that we're interested in. This integral gives us the area. And just a couple of things to notice about it. So t is a constant. It's just fixed. So cosh t over sinh t, which we could also, if we wanted to, this is the hyperbolic cotangent. But that's really not important at all. But we could call it that, if we wanted to. So this is just a constant. So we have minus a constant times y dy. So this part's easy to integrate.

So the hard part is going to be integrating this y squared plus 1. Now, one thing you've seen is that when you have a y squared, a square root of-- when you have y y squared plus 1, one substitution that sometimes works is a tangent substitution. And the reason a tangent substitution works, is that you have a trig identity, tan squared plus 1 equals secant squared.

In this case, I'd like to suggest a different substitution. All right? So this integral is the integral that you want. And I'd like to suggest a substitution, which is that you use a hyperbolic trig function as the thing that you substitute.

So in particular, instead of using, instead of relying on the trig identity tan squared plus 1 equals secant squared, you can use the hyperbolic trig identity, which is that sinh squared u plus 1 equals cosh squared u.

So this identity-- so here we have a something squared plus 1 equals something squared. So the identity this suggests is to try the substitution y equals sinh u. All right? So this is a hyperbolic trig substitution.

So why don't you take that hint, try it out on this integral, see how it goes. Take some time, pause the video, work it out, come back and we can work it out together.

Welcome back. Hopefully you had some luck solving this integral using a hyperbolic trig substitution. Let's work it out together, see if my answer matches the one that you came up with.

So as I said before, this integral comes in two parts. There's the hard part, the square root of y squared plus 1 part, and there's the easy part, this y part. So before I make the substitution, let me just deal with the easy part. So I'll do that over here.

So we have the one part of the area, or one part of the integral, really, is the integral from 0 to sinh t of minus cosh t over sinh t times y dy. OK. So this is just a constant, so it's a constant times y. So this is equal to-- well, it's the same constant comes along, minus cosh t over sinh t, times y squared over 2, for y between 0 and this upper bound, sinh t. So that's equal-- OK. So when y is 0, this is just 0. So this equals minus cosh t sinh t over 2.

So that's the easy part of integral. So in order to compute the total area, we need to add this expression that we just computed to the integral of this first part. So that's what we need to compute next. And that's what we're going to use the hyperbolic trig substitution on.

So we're going to compute the integral from 0 to sinh t of the square root of y squared plus 1 dy. And we're going to use the substitution sinh u equals y, or y equals sinh u. OK. So we need, what do I need? I need what dy is, and I need to change the bounds. So dy-- I'm sorry, I'm going to flip this around to take the-- so dy is, I need the differential of sine u-- sorry, of sinh u. And so we saw in the earlier hyperbolic trig function recitation that that's cosh u du, or if you like, you could just differentiate using the formulas that we know for sinh and cosh.

And we need bounds. So when y is 0, we need sinh of something is 0. And so it happens that that value is 0. So if you remember the graph of the function, or you can just check in the formula, when sinh is 0, when t is 0, that's when you get sinh is 0. It's the only time e to the t equals e to the minus t.

OK. So when y is 0, then u is 0, and when y is sinh t, then u is t. Right? Because sinh u is sinh t.

So under the substitution, this becomes the integral from 0 to t now, from u equals 0 to t, of-- well, OK. So this becomes the square root of sinh squared u plus 1, and then dy is times cosh u du.

OK. Now the reason we made this substitution in the first place is that this, we can use a hyperbolic trig identity here. So sinh squared u plus 1 is just cosh squared u, and square root of cosh squared u is cosh u. Remember that cosh u is positive, so we don't have to worry about an absolute value here. So this is the integral from 0 to t of cosh squared u du.

OK. So at this point, there are a couple of different things you can do. One is that you can, just like when we have certain trig identities, we have corresponding hyperbolic trig identities that we could try out here. So we could try something like that. Another thing you can do, is you can just go back to the formula, right? Cosh t has a simple formula in terms of exponentials, so you can go back to this formula and you can plug in.

So let's just try that quickly, because that's a sort of easy way to handle this. So this is cosh squared u du. So I'm going to write-- OK. Carry that all the way up here. So this is the integral from 0 to t. Well, if you take the formula for hyperbolic cosine and square it, what you get, I'm going to do this all in one step, is you e to the 2u plus 2 plus e to the minus 2u over 4 du.

OK. And so now this is, once you've replaced everything with exponentials, this is easy to integrate. This is-- so e to the 2u, the integral is e to the 2u over 2, so that comes over 8. 2 over 4, you integrate that, and that's just 2u over 4, which is u over 2. And now the last one is minus e to the minus 2u over 8 between 0 and t.

OK. So now we take the difference here. At t, we get e to the 2t over 8 plus t over 2 minus e to the minus 2t over 8. Minus-- OK. And when u is equal-- so that was at u equals t. At u equals 0, we get 1/8 plus 0 minus 1/8. So that's just 0. OK. So this is what we got for that part of the integral.

So OK, so we've now split the integral into two pieces. We computed one piece, because it was just easy, we're integrating a polynomial. We computed the other piece, which was more complicated, using a hyperbolic trig substitution. The whole integral is the sum of those two pieces. So now the whole integral, I have to take this piece, and I have to add it to the thing that I computed for the other piece before, which was somewhere-- where did it go? Here it is, right here. Which was minus cosh t sinh t over 2. OK. So I'm going to save you a little arithmetic, and I'm going to observe that minus cosh t sinh t over 2 is exactly equal to the minus e to the 2t over 8, plus e to the minus 2t over 8. So adding these two expressions together gives us-- so the first expression, minus cosh t sinh t over 2, is minus e to the 2t over 8 plus e to the minus 2t over 8, plus-- OK. Plus what we've got right here, which is e to the 2t over 8 minus e to the minus 2t over 8 plus t over 2. And that's exactly equal to t over 2.

OK. So this is the area of that sort of hyperbolic triangle thing that we started out with at the beginning. So let me just walk back over there for a second.

So we used this hyperbolic trig substitution in order to compute that the area of this triangle is t over 2. And I just want to-- first of all, I want to observe that that's a really nice answer. So that's kind of cool. The other thing that I want to observe is that this is a very close analogy with something that happens in the case of regular circle trigonometric functions. Which is, if you look at a regular circle, and you take the point cosine theta comma sine theta, then the area of this little triangle here is theta over 2.

So in this case, u doesn't measure an angle, but it does measure an area in exactly the same way that theta measures an area. So there's a really cool relationship there between the hyperbolic trig function and the regular trig function. So that's just a kind of cool fact. The useful piece of knowledge that you can extract from what we've just done, though, is that you can use this hyperbolic trig substitution in integrals of certain forms. So in the same way that trig substitutions are suggested by certain forms of the integrand, hyperbolic trig substitutions are also suggested by certain forms of the integrand, and often you have a choice about which one to use. And in this particular instance, a hyperbolic trig substitution worked out quite nicely. Much more nicely than a trig substitution would have worked out.

So it's just another tool for your toolbox. I'll end with that.