

PROFESSOR: Hi. Welcome back to recitation. Today we're going to do a couple of exercises on computing average values and probabilities using integration. So for this problem, I'm going to let R be the name of the region that's bounded above by the curve, by the line I guess, y equals x and below by the curve y equals x cubed. So I have a little picture of R here. It, you know, lives all in the first quadrant there. It's just this little sliver.

So the first part of the question is, if I look at all the points in R , what's that average x -coordinate of those points? What's the average value of the x -coordinate? And the second problem is, if I choose a point at random somewhere in R , what's the probability that its x -coordinate is larger than $1/2$? So what's the probability that it lies on the right-hand side of R ? So why don't you pause the video, take a couple minutes to work through these problems, come back, and we can work on them together.

Alright, welcome back. So hopefully you've had a chance to get some good work done on these questions. So let's start to work through them. So remember, to compute the average value of a function over some region what you need to do is you need to compute a weighted average. And the weighting here is that you have to consider the fact that, you know, for x very near zero, there aren't very many points in R in this little corner. Right? And for x very near 1, there aren't very many points there either. In the middle, this region is a little higher, so there are more points there. So those points will sort of weigh more when you take the average of all points, than the points near the edges.

So the way we account for that is we have this weight function that is the, in our case, since we're interested in the x -coordinate, the weight at a given x -coordinate is the slice of the-- how much of the region lies above that x -coordinate. What, you know, the area of a little rectangle above that x -coordinate says, tells you how many of the points have that x -coordinate. So then we want to average the function x , right? Because we're interested in the x -coordinate, so the function we're averaging is x . Over this region, with that weighting.

So, all right, so let's write down what that means. So we have-- want average of the function f of x , and f of x , the thing we're computing the average of is, just the x -coordinate, so it's just the function value x . Over R . So we want the average of this function, f of x , over R . So what we need to compute is the-- so we have two integrals we need to compute. We need one integral that is the numerator, and so that numerator-- so I'm going to just write average for

the average that we want. So the numerator is the integral-- OK, and so we have to integrate, we have to take all possible x -values into consideration. So x going, in this case that's x going from 0 to 1 and now we want to multiply the function that we're averaging, which in this case is x , by the appropriate weight function. And the weight function is how much of the region is associated with that x -value. And that's the height of this little rectangle, which in this case is x minus x cubed dx .

OK. But then, this is an average, we have to divide by the total weight of the region. The weight, in this case, is just the area. So we have to divide by the integral from 0 to 1 of just this x minus x cubed dx . OK, so we have to compute these two integrals and then we have to take their ratio. So let's do them separately.

So the first one is-- well, the second one is actually a little simpler. The one in the denominator, so let's handle that first. The integral from 0 to 1 of x minus x cubed dx . Well, this is a pretty easy integral. It's x squared over 2 minus x to the fourth over 4 between 0 and 1. So that's $1/2$ minus $1/4$ minus, well when you put in 0 you just get 0. So that's $1/4$.

And the first one, the top, the numerator is the integral from 0 to 1. OK, we can multiply through, so that's x squared minus x to the fourth dx . So that integral from 0 to 1 again is x cubed over 3 minus x to the fifth over 5 between 0 and 1. When we put in 0 we get 0, put in 1 we get $1/3$ minus $1/5$. So that's common denominator 15. $2/15$. So the numerator of our average value is $2/15$, the denominator is $1/4$. So the average value we're interested in is $2/15$ divided by $1/4$, which is $8/15$.

So the average x -coordinate is just, so $8/15$ is just a tiny bit larger than $1/2$. So this is saying somehow the average x -coordinate is just slightly shifted to the right of $1/2$. So this, I may have drawn this region sort of symmetrically, but in fact it's a little bit shifted to the right there. So that's the the first part of the problem.

For the second part, we want to compute the probability-- so OK, so we choose a point at random in this set R and we want to know what's the probability that its x -coordinate is larger than $1/2$. Well, since all points are, you know, equally likely, all regions, the probability just has to do with the area of the region. What we really want to know is, what's is area of R to the right of the line x equals $1/2$. So that's the good region, and then we want to know how much of the entire region is that. So we want to know the area of the good region to the right of $1/2$ divided by the total area.

Now luckily we've already computed the total area, right here. This was this denominator that we had over here. So we just need the numerator of that fraction. So let's go over here, let me write that down. So the probability equals-- I'm going to put good area in quotation marks-- so what I mean by good area is the area of the set of points that satisfies our condition. And in our case, the good area is just-- so a point we're interested in, if its x-coordinate is bigger than $1/2$, so we just want to take the part of this region to the right of $1/2$. So in order to compute that, we just take this integral from $1/2$ to 1 instead of from 0 to 1 . So we're only counting those points to the right of $1/2$ and then we want the, you know, the area between these two curves on that region.

OK, so, and again this is a fairly simple integral to compute. So this gives me x^2 over 2 minus x to the fourth over 4 between $1/2$ and 1 . All right, so this is maybe a tiny bit hairy, so this gives me $1/2$ minus $1/4$ minus-- well we put in $1/2$ here we get $1/8$ minus-- $1/2$ to the fourth is $1/16$, divided by 4 is $1/64$. So this is all going to be in sixty-fourths.

So OK. So let's see, $32/64$ minus $16/64$ minus $8/64$ plus 1 . If I just put it all over that common denominator. All right, so I think that looks like $9/64$, if I did that right. So, OK. So that's the good area. And then the probability that I want, I have to divide the good area by the total area. And we saw before that the total area was $1/4$. And that was what this computation was. So the total area's $1/4$, so the probability-- I'm just going to write pr for probability-- that we're interested in is $9/64$ divided by $1/4$ which is $9/16$.

OK, so this says-- also $9/16$ is also a little bit bigger than a half. So this is a different way of saying, our region, there's a little bit more of it to the right than there is to the left. You know, so it's slightly more likely that a random point is to the right of the line y equals $1/2$ than it is to the left of that line.

So just to sum up, we had these two different problems that we did concerning this region R . So first we computed the average value of the x-coordinate of a point in this region. So that was part a. We computed the average value of the x-coordinate, and that was over here. So we had to, you know, do this, the weighted average of the x-coordinate divided by the total area. And then second, we computed the probability that a randomly chosen point has x-coordinate larger than $1/2$. And we did that over here. And for that we needed to compute the area of the good region, which was the part to the right of the line y equals $1/2$ and then we needed to divide it by the total area. So I'll end there.