

Example: Boy Near a Dart Board

Suppose a seven year old child is throwing darts at a dartboard while her little brother is standing nearby. What is the probability that the brother gets hit by a dart?

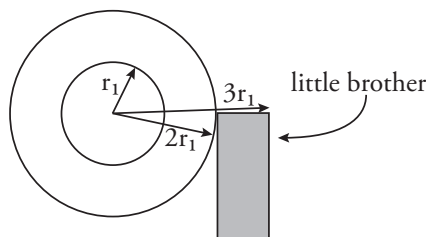


Figure 1: A younger brother stands by a dart board.

In order to turn this into a math problem we must make some assumptions. First, we'll assume something about her aim:

$$\text{Number of hits} = ce^{-r^2}$$

This says that the locations she hits are normally distributed (like a bell curve) about the center of the dart board. She's more likely to hit the dart board, but there's some chance she'll hit the wall next to it, or her brother.

As usual, we'll compute the probability by finding the ratio of the part to the whole. For the final calculation, the "part" will be where the brother is, and the "whole" is all the possible places the dart could hit.

We start by considering a thin ring or annulus of the dart board. The inside of the ring will have radius r_1 and the outside will have radius r_2 .

If we graph $y = e^{-r^2}$ we get a "side view" of our probability distribution that looks like Figure 3. The height of the graph indicates that the darts are most likely to land near the center of the dartboard and less likely to land further out. (It turns out that the c in ce^{-r^2} will cancel, so we'll forget about it for now. The value of c depends on the number of darts thrown.)

To calculate the probability of a dart landing in the annulus described by r_1 and r_2 we look at the area between r_1 and r_2 , above the r -axis, and below $y = e^{-r^2}$. We'll calculate the volume of revolution of this area, revolving about the center of the dart board (which corresponds to the y -axis in our graph). This volume of revolution will have a ring or washer shape.

Since we've written probability as a function of radius, we'll use the method of shells; the probability of a hit is constant at a given radius. To find the probability of hitting inside the annulus, we use the limits of integration r_1 and

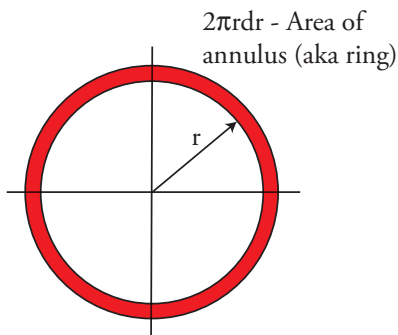


Figure 2: A annulus or ring about the center of the dartboard.

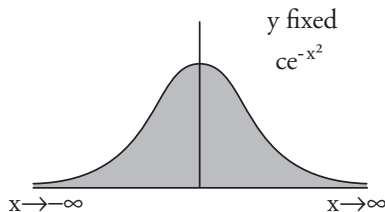


Figure 3: Graph of ce^{-x^2} .

r_2 , circumference $2\pi r$, and height e^{-r^2} . We're integrating with respect to r so the thickness of a shell is dr .

$$\int_{r_1}^{r_2} (2\pi r) e^{-r^2} dr$$

Question: Why not just find the area under the curve between r_1 and r_2 , then revolve that?

Answer: Because a shell with radius r_1 may have significantly less volume than one with radius r_2 . We can't simply multiply that area by 2π to get an estimate of the volume of the ring.

Question: So r_1 and r_2 could be anything?

Answer: Yes. We'll do this in general first, and then later we can plug in values for r_1 and r_2 .

Question: Why do we have to compute the volume?

Answer: We're computing the number of hits per unit area. The height of the graph corresponds to the number of hits, so we're computing area times height equals volume.

Question: Why is this a realistic calculation?

Answer: The function e^{-r^2} turns out to be the most accurate one for describing where the darts are likely to hit; this model was used to analyze where the V-2 rockets from Germany hit London during World War II.

We're constructing a mathematical model of dart throwing. The probability function e^{-r^2} models the fact that the kid is aiming for the center of the dartboard, but that due to inaccuracies in her throwing she won't always hit the bullseye. The assumption that the model describes the situation accurately does need to be justified, but for now Professor Jerison is asking us to accept it as-is.

Now we'll calculate the probability of a dart landing in the annulus between radii r_1 and r_2 :

$$\begin{aligned}\int_{r_1}^{r_2} (2\pi r)e^{-r^2} dr &= -\pi e^{-r^2} \Big|_{r_1}^{r_2} \\ &= -\pi e^{-r_2^2} - (-\pi e^{-r_1^2}) \\ &= \pi(e^{-r_1^2} - e^{-r_2^2}).\end{aligned}$$

Note that the calculation doesn't change much if we instead use ce^{-r^2} as the probability function. We'll put it back now:

$$\text{Part} = c\pi(e^{-r_1^2} - e^{-r_2^2})$$

To calculate the volume of the "Whole", we let r range from zero to infinity. This is slightly artificial because when you're playing darts you don't hit the floor or the ceiling. But for the same reason, pretending the wall goes on forever doesn't have much effect on our final answer. The easiest value to calculate has the upper limit of integration equal to infinity; using this approximation will make the numbers come out more cleanly and because the e^{-r^2} is close to zero when r is large our answer will still be pretty accurate.

$$\text{Whole} = c\pi(e^{-0^2} - e^{-\infty^2}) = c\pi(1 - 0) = c\pi$$

The probability of hitting the annulus is then:

$$\begin{aligned}P(r_1 < r < r_2) &= \frac{\text{Part}}{\text{Whole}} \\ &= \frac{c\pi(e^{-r_1^2} - e^{-r_2^2})}{c\pi} \\ P(r_1 < r < r_2) &= e^{-r_1^2} - e^{-r_2^2}.\end{aligned}$$

(Notice that the c did, in fact, cancel.)

For this to be a realistic probability function, it should be true that

$$P(0 < r < \infty) = 1.$$

Since $e^0 = 1$ and $e^{-\infty} = 0$, this is in fact true. Another realistic assumption is that the probability of the child hitting the target is about $1/2$. In other words, if a is the radius of the target,

$$P(0 \leq r \leq a) = \frac{1}{2}.$$

To figure out the probability of hitting the little brother, we have to define a “part” corresponding to the little brother. Let’s suppose he’s not standing too close — maybe he’s standing a distance of $2a$ away from the dart board. Volumes of revolution are much easier to compute than volumes of preschoolers, so we’ll assume that the parts of the little brother that are most likely to be hit lie on an arc of an annulus. We’ll say that he’s between $2a$ and $3a$ units away from the dartboard, and that he fills the angular arc between 3 o’clock and 5 o’clock to the right of the board, which is $1/6$ of a circle.

So the probability of the little brother getting hit is:

$$\frac{1}{6}P(2a < r < 3a).$$

In order to get a number out of this we need to use the fact that the probability of the girl hitting the dart board is $1/2$ or 50%.

$$\begin{aligned} P(0 < r < a) &= 1/2 \\ e^{0^2} - e^{-a^2} &= 1/2 \\ 1 - e^{-a^2} &= 1/2 \\ 1/2 &= e^{-a^2}. \end{aligned}$$

We could calculate a from this, but it turns out we don’t need to. We now know all we need to calculate the probability of the little brother being hit. We start by calculating the probability of hitting the entire annulus between $2a$ and $3a$.

$$\begin{aligned} P(2a < r < 3a) &= e^{-(2a)^2} - e^{-(3a)^2} \\ &= e^{-2^2 a^2} - e^{-3^2 a^2} \\ &= e^{-4a^2} - e^{-9a^2} \\ &= (e^{-a^2})^4 - (e^{-a^2})^9 \\ &= \left(\frac{1}{2}\right)^4 - \left(\frac{1}{2}\right)^9 \\ &= \frac{1}{16} - \frac{1}{512} \\ &= \frac{1}{16} - \frac{1}{512} \\ &\approx \frac{1}{16}. \end{aligned}$$

(Getting the value $\frac{1}{512}$ for $r = 3a$ reassures us that the probably of getting hit does indeed decrease sharply as you get further from the dart board.)

So the probability that the girl hits her little brother is:

$$\begin{aligned}\frac{1}{6}P(2a < r < 3a) &\approx \frac{1}{6} \frac{1}{16} \\ &\approx \frac{1}{100} \\ &= 1\%.\end{aligned}$$

In this problem, our weight function was:

$$w(r) = 2\pi cr e^{-r^2}.$$

This is different from the ce^{-r^2} that we started out with. That function is one dimensional; to take into account that we're describing the probability for a whole ring around the center of the dart board we include a factor of $2\pi r$.

If you graph this function you'll see that as r goes to 0 the value of $w(r)$ also goes to zero. This reflects the fact that it's hard to hit the center of the target because the center is very small. It's more likely that a dart will hit a ring around the center, because those rings have greater area than the bull's eye.

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