1. (a) Give a general expression for the quadratic approximation to a twice differentiable function $f(x)$ at $x = a$.

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2$$

(b) Use your answer from part (a) to give an approximate value for $\ln(1.2)$, where $\ln(x)$ is the natural log function.

$$(\ln x)' = \frac{1}{x}, \qquad (\ln x)'' = -\frac{1}{x^2}$$

$$\begin{aligned}
\ln(1.2) &\approx \ln 1 + (\ln'(1))(1.2-1) + \frac{\ln''(1)}{2}(1.2-1)^2 \\
&= 0 + 0.2 - \frac{1}{2}(0.2)^2 \\
&= 0.2 - \frac{0.04}{2} \\
\ln(1.2) &\approx 0.18
\end{aligned}$$

2. Salt is poured from a conveyer belt at a rate of 30 ft^3/min, forming a conical pile with a circular base whose height and diameter of base are always equal. How fast is the height of the pile increasing when the pile is 10 ft. high?

$$\begin{aligned}
h &= 2r \\
V &= \frac{1}{3}\pi r^2 h \\
&= \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h \\
&= \frac{1}{12}\pi h^3 \\
\frac{dV}{dt} &= \frac{1}{4}\pi h^2 \frac{dh}{dt} \\
\Rightarrow \frac{dh}{dt} &= \frac{dV}{dt}\left(\frac{4}{\pi h^2}\right) \\
&= (30)\left(\frac{4}{10^2\pi}\right)
\end{aligned}$$

$$= \frac{120}{100\pi}$$
$$= \frac{12}{10\pi}\frac{\text{ft}}{\text{min}}$$

3. Draw a careful picture of the graph of the function

$$f(x) = x - 3x^{1/3}.$$

Be sure to indicate the coordinates of any local maxima and minima, the intervals on which the function is increasing and decreasing, and asymptotes (if any of these features occur). Computing inflection points may help you draw an accurate picture, but is not necessary.

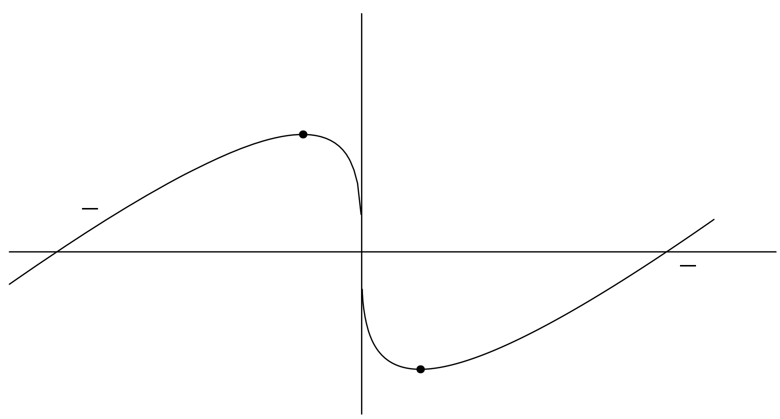
When $f(x) = 0$, $x - 3x^{1/3} = 0$, so $x^{1/3}(x^{2/3} - 3) = 0$.
$f(x) = 0$ when $x = 0, 3\sqrt{3}, -3\sqrt{3}$.

$f'(x) = 1 - 3 \cdot \frac{1}{3}x^{-2/3} = 1 - x^{-2/3}$: critical points are at $x = \pm 1$.

Critical points are $(1, -2)$ (minimum) and $(1, 2)$ (maximum).

The tangent line to the graph is vertical at $x = 0$.

$f(x)$ is increasing on $(-\infty, -1)$ and $(1, \infty)$ and decreasing on $(-1, 1)$.



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4. A metal storage tank is to be made in the shape of a cylinder with a circular base and a hemispherical top. Find the dimensions of the tank which require the least amount of metal used to hold a fixed constant volume V.

$(-1, 2)$

$-3\sqrt{3}$

Start by drawing a diagram.

Constant $V = \frac{1}{2}\left(\frac{4}{3}\pi r^3\right) + \left(\pi r^2 h\right) = \frac{2}{3}\pi r^3 + \pi r^2 h$.

$3\sqrt{3}$ x

$(1, -2)$

(Assuming) the amount of metal used is proportional to the surface area, we want to minimize surface area SA.

$$\begin{aligned} SA &= \frac{1}{2}(4\pi r^2) + (2\pi r)h + \pi r^2 \\ &= 2\pi r^2 + 2\pi rh + \pi r^2 \\ SA &= 3\pi r^2 + 2\pi rh \end{aligned}$$

$$h = \frac{V - \frac{2}{3}\pi r^3}{\pi r^2}$$

$$\begin{aligned} SA &= 3\pi r^2 + 2\pi r\left(\frac{V - \frac{2}{3}\pi r^3}{\pi r^2}\right) \\ &= 3\pi r^2 + \frac{2V}{r} - \frac{4}{3}\pi r^2 \\ &= \frac{5}{3}\pi r^2 + \frac{2V}{r}. \end{aligned}$$

$\frac{d}{dr}SA = \frac{10}{3}\pi r + (-1)\frac{2V}{r^2}$. Set this equal to 0, assuming $r \neq 0$.

$$\begin{aligned} \frac{10}{3}\pi r + (-1)\frac{2V}{r^2} &= 0 \\ \frac{5}{3}\pi r &= \frac{V}{r^2} \\ V &= \frac{5}{3}\pi r^3 \end{aligned}$$

This tells us that $r = \sqrt[3]{\frac{3V}{5\pi}}$ and:

$$h = \frac{\frac{5}{3}\pi r^3 - \frac{2}{3}\pi r^3}{\pi r^2} = \frac{\pi r^3}{\pi r^2} = r.$$

$h = r = \sqrt[3]{\frac{3V}{5\pi}}$ is a critical point. Show that it is indeed a minimum by checking the boundaries $r = 0$ and $h = 0$.

When $r = 0$, $SA = 0 + \infty$. This is not a minimum.

When $h = 0$, $SA = 3\pi r^2$ and $V = \frac{2}{3}\pi r^3$. Therefore $r = \sqrt[3]{\frac{3V}{2\pi}}$ and $SA = 3\pi\left(\frac{3V}{2\pi}\right)^{2/3}$ when $h = 0$.

When $h = r = \sqrt[3]{\frac{3V}{5\pi}}$, $SA = 5\pi\left(\frac{3V}{5\pi}\right)^{2/3}$.

Comparing the surface areas when $h = r$ and when $h = 0$ by cubing both sides, we see that:

$$125 \cdot 9/25 = 45 < 27 \cdot 9/4 = 60.75.$$

We conclude that the critical point represents the minimum surface area, and so the answer to the question is $\boxed{h = r = \sqrt[3]{\frac{3V}{5\pi}}}$

5. Explain why Newton's method eventually fails when finding zeroes of $f(x) = x^3 - 3x + 7$ with a starting value $x_1 = 2$.

   Newton's method says that $x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})}$. Here $f'(x) = 3x^2 - 3$.

   For $x_1 = 2$, $x_2 = 2 - \frac{f(2)}{f'(2)} = 2 - \frac{9}{9} = 1$.

   $x_3 = 1 - \frac{f(1)}{f'(1)} = 1 - \frac{5}{0}$ is undefined.

   In particular, $f'(x_2) = 0$.

   The problem is that the tangent line to the graph of f at $x = x_2 = 1$ is horizontal. Thus, it does not intersect the x-axis and we cannot continue Newton's method.

6. Prove that
$$\sqrt{1+x} < 1 + \frac{1}{2}x, \quad \text{if } x > 0.$$

   Let $g(x) = 1 + \frac{1}{2} - \sqrt{1+x}$. We need to show that $g(x) > 0$ for $x > 0$.

   $g(0) = 1 + 0 - 1 = 0$, so by the mean value theorem, it's sufficient to show that $g'(x) > 0$ for all $x > 0$.

$$g'(x) = 0 + \frac{1}{2} - \frac{1}{2}\frac{1}{\sqrt{1+x}} = \frac{1}{2} - \frac{1}{2\sqrt{1+x}}.$$

   Because $x > 0$,

$$\begin{aligned} \sqrt{1+x} &> \sqrt{1} = 1, && \text{so} \\ \frac{1}{\sqrt{1+x}} &< \frac{1}{1} = 1, && \text{and} \\ -\frac{1}{2\sqrt{1+x}} &> -\frac{1}{2} \cdot 1 = -\frac{1}{2}. \end{aligned}$$

   Therefore $g'(x) = \frac{1}{2} - \frac{1}{2\sqrt{1+x}} > \frac{1}{2} - \frac{1}{2} = 0$.

   We have shown $g(0) = 0$ and $g'(x) > 0$ for $x > 0$, so $g(x) > 0$ for $x > 0$, which implies $\sqrt{1+x} < 1 + \frac{1}{2}x$ if $x > 0$.