Practice Exam 3 Solutions

Problem 1. Evaluate $\int \frac{t^3+t}{\sqrt{1+t^2}} dt$

$\boxed{\text{Solution}}$ The problem can be simplified as $t^3 + t = t(t^2 + 1)$. Then by substitution with $u = t^2 + 1$ and thus $du = 2t\ dt$ we have

$$\int t\sqrt{t^2+1}\ dt = \frac{1}{2}\int u^{1/2} du = \frac{1}{3}u^{3/2} + C = \frac{1}{3}(t^2+1)^{3/2} + C.$$

Problem 2. Evaluate $\int_3^5 x^3\sqrt{x^2-9}dx$

$\boxed{\text{Solution}}$ We begin by making the substitution $x^2 - 9 = u$. Then $2xdx = du$ and $x^2 = u + 9$. Substituting in, we get

$$\int_0^{16} (u+9)\sqrt{u}\frac{du}{2} = \frac{1}{2}\int_0^{16} u^{3/2} + 9u^{1/2} du = \frac{1}{2}\left(\frac{2}{5}u^{5/2} + \frac{18}{3}u^{3/2}\right)\Bigg|_0^{16} = \frac{1}{5}16^{5/2} + 3\cdot 16^{3/2}.$$

Problem 3: Suppose that $\lim_{x\to a^+} g(x) = B \neq 0$ where B is finite and $\lim_{x\to a^+} h(x) = 0$, but $h(x) \neq 0$ in a neighborhood of a. Prove that

$$\lim_{x\to a^+}\left|\frac{g(x)}{h(x)}\right| = \infty.$$

$\boxed{\text{Solution}}$ Let $M \in \mathbb{R}^+$ and set $\epsilon = 1/(2M) > 0$. By hypothesis, there exist δ_1, δ_2 such that $|g(x) - B| < |B|/2$ if $0 < x - a < \delta_1$ and $|h(x)| < |B|\epsilon$ if $0 < x - a < \delta_2$. Choose $\delta = \min\{\delta_1, \delta_2\}$. Then for $0 < x - a < \delta$, $|g(x)| > |B|/2$ and $|h(x)|^{-1} > 1/(|B|\epsilon)$. Thus

$$\frac{|g(x)|}{|h(x)|} > \frac{|B|}{2|B|\epsilon} = \frac{1}{2\epsilon} = M.$$

This proves the result.

Problem 4. Let $f(x) : [0, \infty) \to \mathbb{R}^+$ be a positive continuous function such that $\lim_{x\to\infty} f(x) = 0$. Prove there exists $M \in \mathbb{R}^+$ such that $\max_{x\in[0,\infty)} f(x) = M$.

$\boxed{\text{Solution}}$ By hypothesis, there exists $N \in \mathbb{R}^+$ such that for all $x > N$, $f(x) < f(1)$. (We don't need absolute values here as f is positive.) Now consider the interval $[0, N]$. As f is continuous and $[0, N]$ is closed, the Extreme Value Theorem tells us there exists $w \in [0, N]$ such that $f(w) \geq f(x)$ for all $x \in [0, N]$. That is, $f(w) \geq f(1)$. As $f(1) > f(x)$ for all $x > N$, $f(w) \geq f(x)$ for all $x \in [0, \infty)$.

Problem 5.

- A sequence is called *Cauchy* if for all $\epsilon > 0$ there exists $N \in \mathbb{Z}^+$ such that for all $m, n > N$, $|a_m - a_n| < \epsilon$. Prove that if $\{a_n\}$ is a convergent sequence, then it is Cauchy. (The converse is also true.)

- A function $f : \mathbb{R} \to \mathbb{R}$ is called a *contraction* if there exists $0 \leq \alpha < 1$ such that $|f(x) - f(y)| \leq \alpha|x - y|$. Let f be a contraction. For any $x \in \mathbb{R}$, prove the sequence $\{f^n(x)\}$ is Cauchy, where $f^n(x) = f \circ f \circ \cdots \circ f(x)$ (the n times composition of f with itself).

$\boxed{\text{Solution}}$ (a) By hypothesis $\{a_n\}$ is a convergent sequence, with limit L. Let $\epsilon > 0$. Then there exists $N \in \mathbb{Z}^+$ such that for all $n \geq N$, $|a_n - L| < \epsilon/2$. Thus, for all $m, n \geq N$, $|a_m - a_n| \leq |a_m - L| + |a_n - L| < \epsilon/2 + \epsilon/2 = \epsilon$, where the first inequality follows by the triangle inequality. It follows that $\{a_n\}$ is Cauchy.

(b) Fix $x \in \mathbb{R}$ and denote $|x - f(x)| = C$. We first claim, $|f^n(x) - f^{n+1}(x)| \leq \alpha^n \cdot C$ and proceed to prove it by induction. Notice that $|f(x) - f^2(x)| \leq \alpha|x - f(x)| = \alpha C$ so the statement holds for $n = 1$. Now assume the statement holds for some n. We proceed to show it holds for $n + 1$. As f is a contraction, and by the induction hypothesis

$$|f^{n+1}(x) - f^{n+2}(x)| = |f(f^n(x)) - f(f^{n+1}(x))| \leq \alpha|f^n(x) - f^{n+1}(x)| \leq \alpha\alpha^n C = \alpha^{n+1}C.$$

Now consider the quantity $|f^n(x) - f^k(x)|$ for $k > n$. Notice, by the triangle inequality,

$$|f^n(x) - f^k(x)| \leq \sum_{i=0}^{k-1} |f^{n+i}(x) - f^{n+i+1}(x)| \leq C \sum_{i=0}^{k-1} \alpha^{n+i} = C\alpha^n \sum_{i=0}^{k-1} \alpha^i.$$

As $0 \leq \alpha < 1$, $\sum_{i=0}^{k-1} \alpha^i < \frac{1}{1-\alpha}$ for all k. Now, for any $\epsilon > 0$, choose N such that $\frac{C}{1-\alpha} \cdot \alpha^N < \epsilon$. Then for any $n, k \geq N$, the previous work implies

$$|f^n(x) - f^k(x)| \leq \frac{C}{1 - \alpha} \cdot \alpha^{\min\{n,k\}} \leq \frac{C}{1 - \alpha} \cdot \alpha^N < \epsilon.$$