1.21 Integrability of bounded piecewise-monotonic functions.

The definition of "piecewise-monotonic" is given on p. 77 of the text.

Lemma. If f is bounded on $[a,b]$ and monotonic on (a,b), then f is integrable on $[a,b]$.

(Note that we need to assume f is bounded in the hypothesis of this lemma. The function

$$f(x) = \begin{cases} 1/x & \text{for} \quad 0 < x \leq 1 \\ 0 & \text{for} \quad x = 0 \end{cases}$$

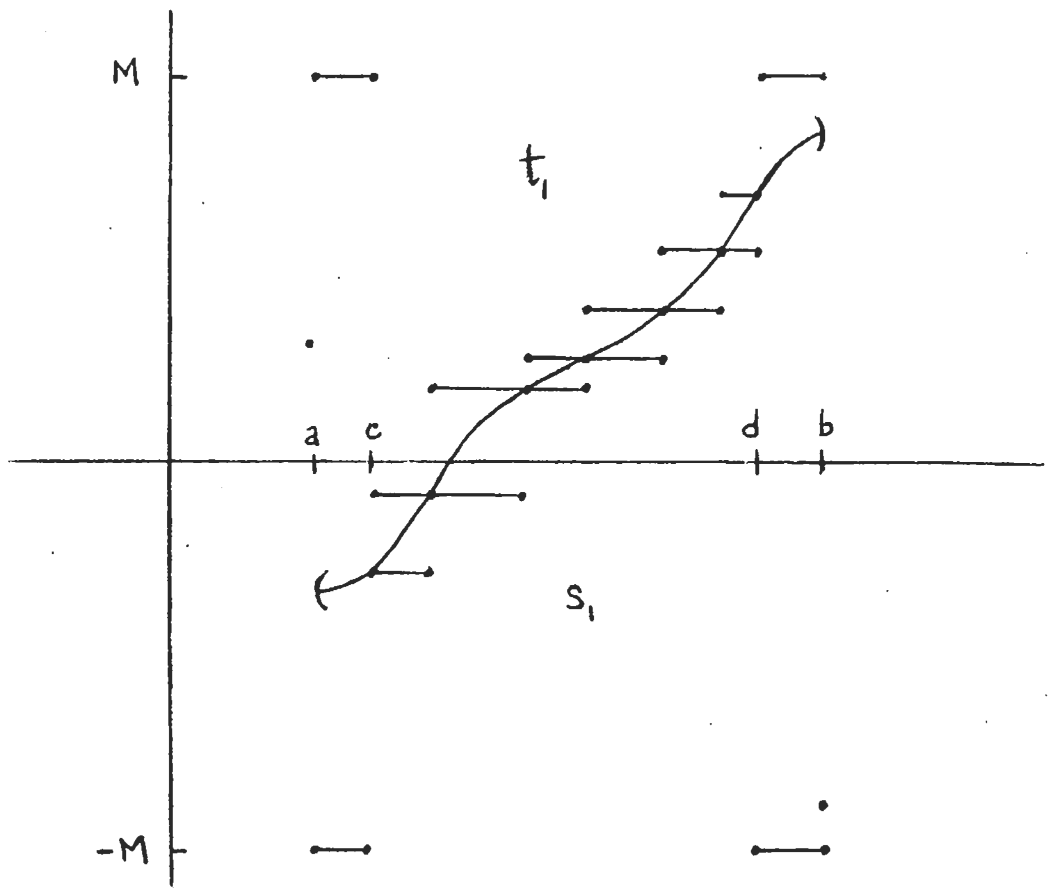
is monotonic on $(0,1)$, but it is not bounded.)

Proof. Choose M so that $-M \leq f(x) \leq M$. We apply the Riemann condition.

Given $\varepsilon > 0$, let us choose numbers c and d (close to a and b respectively), such that

$$a < c < d < b$$

and such that $c - a < \varepsilon/M$ and $b - d < \varepsilon/M$.



Now f is monotonic on $[c,d]$ so it is integrable on $[c,d]$. Therefore we can find step functions s and t defined on $[c,d]$ such that $s \leqslant f \leqslant t$ on $[c,d]$, and such that $\int_c^d t - \int_c^d s < \varepsilon$. <u>Extend</u> t to a step function t_1 defined on $[a,b]$ by setting

$$t_1(x) = \begin{cases} M & \text{for} \quad a \leqslant x \leqslant c, \\ t(x) & \text{for} \quad c \leqslant x \leqslant d, \\ M & \text{for} \quad d < x \leqslant b. \end{cases}$$

Similarly, extend s to a step function s_1 defined on $[a,b]$ by setting

$$s_1(x) = \begin{cases} -M & \text{for} \quad a \leqslant x < c, \\ t(x) & \text{for} \quad c \leqslant x \leqslant d, \\ -M & \text{for} \quad d < x \leqslant b. \end{cases}$$

Then $s_1 \leqslant f \leqslant t_1$ on all of $[a,b]$. Furthermore,

$$\int_a^b t_1 - \int_a^b s_1 = \int_a^c (t_1-s_1) + \int_c^d (t_1-s_1) + \int_d^b (t_1-s_1)$$

$$= 2M(c-a) + \int_c^d (t_1-s_1) + 2M(d-b)$$

$$< 2\varepsilon + \varepsilon + 2\varepsilon = 5\varepsilon.$$

Since ε is arbitrary, the Riemann condition is satisfied. □

<u>Theorem</u>. <u>If</u> f <u>is bounded and piecewise-monotonic on</u> $[a,b]$, <u>then</u> f <u>is integrable on</u> $[a,b]$.

<u>Proof</u>. By hypothesis, there is a partition $x_0 < x_1 < \ldots < x_n$ of $[a,b]$ such that f is monotonic on each open interval (x_{i-1},x_i). By the preceding lemma, f is integrable on $[x_{i-1},x_i]$ for each i. By the additivity property of integrals (Theorem on p. D.1), it follows that f is integrable on $[a,b]$. □

<u>Exercise</u>

1. Suppose f is bounded on $[a,b]$. Suppose also that f is integrable on <u>every</u> closed interval $[c,d]$ contained in the open interval (a,b). Show that f is integrable on $[a,b]$.