18.01 Single Variable Calculus
Fall 2006

Lecture 36: Infinite Series and Convergence Tests

Infinite Series

Geometric Series

A geometric series looks like

$$1 + a + a^2 + a^3 + ... = S$$

There's a trick to evaluate this: multiply both sides by a:

$$a + a^2 + a^3 + ... = aS$$

Subtracting,

$$(1 + a + a^2 + a^3 + \cdots) - (a + a^2 + a^3 + \cdots) = S - aS$$

In other words,

$$1 = S - aS \implies 1 = (1 - a)S \implies S = \frac{1}{1 - a}$$

This only works when $|a| < 1$, i.e. $-1 < a < 1$.

$a = 1$ can't work:

$$1 + 1 + 1 + ... = \infty$$

$a = -1$ can't work, either:

$$1 - 1 + 1 - 1 + ... \neq \frac{1}{1 - (-1)} = \frac{1}{2}$$

Notation

Here is some notation that's useful for dealing with series or sums. An infinite sum is written:

$$\sum_{k=0}^{\infty} a_k = a_0 + a_1 + a_2 + ...$$

The finite sum

$$S_n = \sum_{k=0}^{n} a_k = a_0 + ... + a_n$$

is called the "n^{th} partial sum" of the infinite series.

Definition

$$\sum_{k=0}^{\infty} a_k = s$$

means the same thing as

$$\lim_{n\to\infty} S_n = s, \text{ where } S_n = \sum_{k=0}^{n} a_k$$

We say the series *converges* to s, if the limit exists and is finite. The importance of convergence is illustrated here by the example of the geometric series. If $a = 1, S = 1 + 1 + 1 + ... = \infty$. But

$$S - aS = 1 \quad \text{or} \quad \infty - \infty = 1$$

does not make sense and is not usable!

Another type of series:

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

We can use integrals to decide if this type of series converges. First, turn the sum into an integral:

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \sim \int_1^{\infty} \frac{dx}{x^p}$$

If that improper integral evaluates to a finite number, the series converges.

Note: This approach only tells us whether or not a series converges. It does **not** tell us what number the series converges to. That is a much harder problem. For example, it takes a lot of work to determine

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

Mathematicians have only recently been able to determine that

$$\sum_{n=1}^{\infty} \frac{1}{n^3}$$

converges to an irrational number!

Harmonic Series

$$\sum_{n=1}^{\infty} \frac{1}{n} \sim \int_1^{\infty} \frac{dx}{x}$$

We can evaluate the improper integral via Riemann sums.

We'll use the upper Riemann sum (see Figure 1) to get an upper bound on the value of the integral.

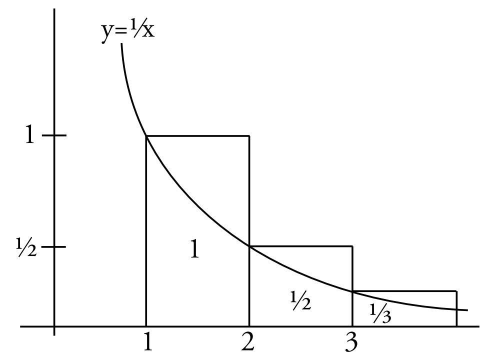

Figure 1: Upper Riemann Sum.

$$\int_1^N \frac{dx}{x} \leq 1 + \frac{1}{2} + ... + \frac{1}{N-1} = s_{N-1} \leq s_N$$

We know that

$$\int_1^N \frac{dx}{x} = \ln N$$

As $N \to \infty$, $\ln N \to \infty$, so $s_N \to \infty$ as well. In other words,

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

diverges.

Actually, s_N approaches ∞ rather slowly. Let's take the lower Riemann sum (see Figure 2).

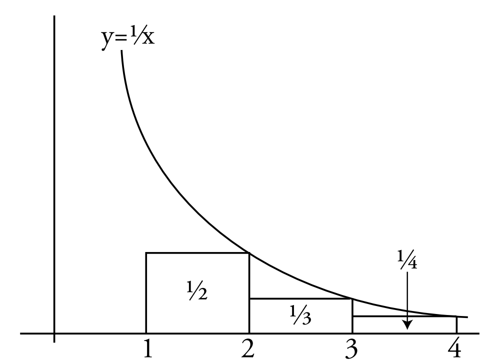

Figure 2: Lower Riemann Sum.

$$s_N = 1 + \frac{1}{2} + ... + \frac{1}{N} = 1 + \sum_{n=2}^{N} \frac{1}{n} \leq 1 + \int_1^N \frac{dx}{x} = 1 + \ln N$$

Therefore,

$$\ln N < s_N < 1 + \ln N$$

Integral Comparison

Consider a positive, decreasing function $f(x) > 0$. (For example, $f(x) = \frac{1}{x^p}$)

$$\left| \sum_{n=1}^{\infty} f(n) - \int_1^{\infty} f(x)dx \right| < f(1)$$

So, either both of the terms converge, or they both diverge. This is what we mean when we say

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \sim \int_1^{\infty} \frac{dx}{x^p}$$

Therefore, $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges for $p \leq 1$ and converges for $p > 1$.

Lots of fudge room: in comparison.

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2 + 10}}$$

diverges, because

$$\frac{1}{\sqrt{n^2 + 10}} \sim \frac{1}{(n^2)^{1/2}} = \frac{1}{n}$$

Limit comparison:
If $f(x) \sim g(x)$ as $x \to \infty$, then $\sum f(n)$ and $\sum g(n)$ either both converge or both diverge.

What, exactly, does $f(x) \sim g(x)$ mean? It means that

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = c$$

where $0 < c < \infty$.

Let's check: does the following series converge?

$$\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^5 - 10}}$$

$$\frac{n}{\sqrt{n^5 - 10}} \sim \frac{n}{n^{5/2}} = n^{-3/2} = \frac{1}{n^{3/2}}$$

Since $\frac{3}{2} > 1$, this series does converge.

Playing with blocks

At this point in the lecture, the professor brings out several long, identical building blocks.

Do you think it's possible to stack the blocks like this?

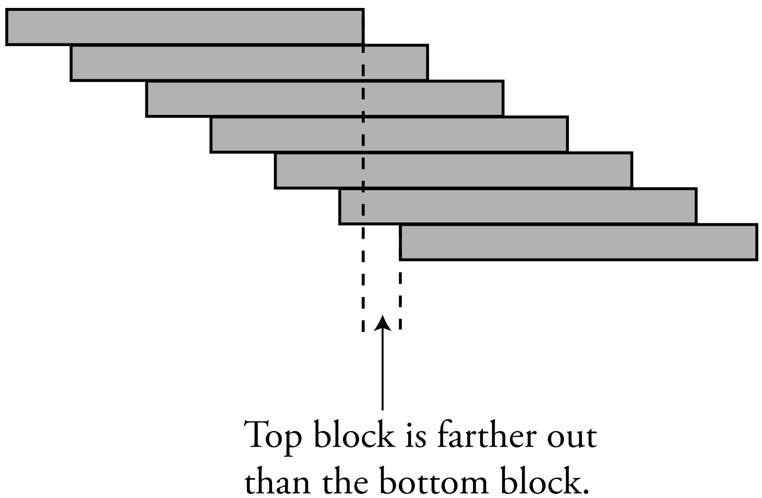

Figure 3: Collective center of mass of upper blocks is always over the base block.

In order for this to work, you want the collective center of mass of the upper blocks always to be over the base block.

The professor successfully builds the stack.

Is it possible to extend this stack clear across the room?

The best strategy is to build from the *top* block down.
Let C_0 be the left end of the first (top) block.
Let C_1 = the center of mass of the first block (top block).
Put the second block as far to the right as possible, namely, so that it's left end is at C_1 (Figure 4).
Let C_2 = the center of mass of the top two blocks.
Strategy: put the *left end* of the next block underneath the center of mass of all the previous ones combined. (See Figure 5).

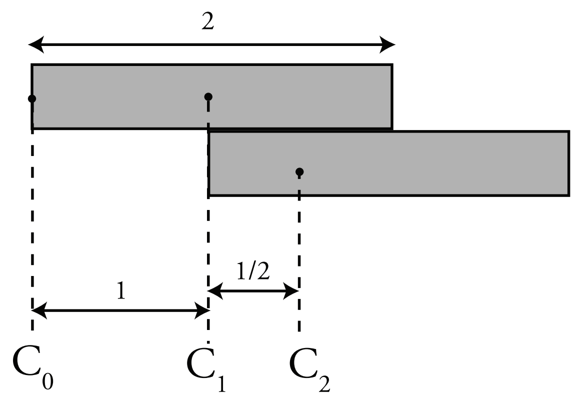

Figure 4: Stack of 2 Blocks.

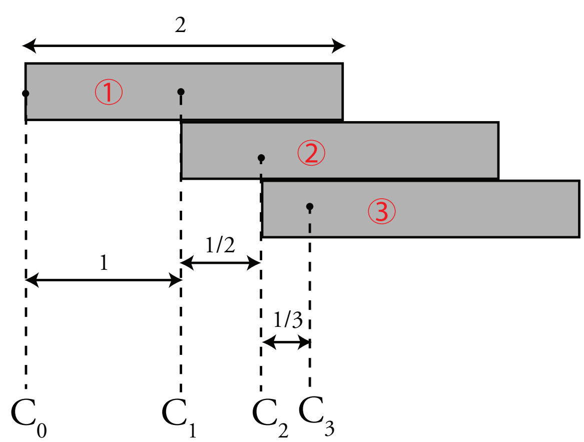

Figure 5: Stack of 3 Blocks. Left end of block 3 is C_2 = center of mass of blocks 1 and 2.

$$C_0 = 0$$
$$C_1 = 1$$
$$C_2 = 1 + \frac{1}{2}$$

$$C_{n+1} = \frac{nC_n + 1(C_n+1)}{n+1} = \frac{(n+1)C_n + 1}{n+1} = C_n + \frac{1}{n+1}$$
$$C_3 = 1 + \frac{1}{2} + \frac{1}{3}$$
$$C_4 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}$$
$$C_5 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} > 2$$

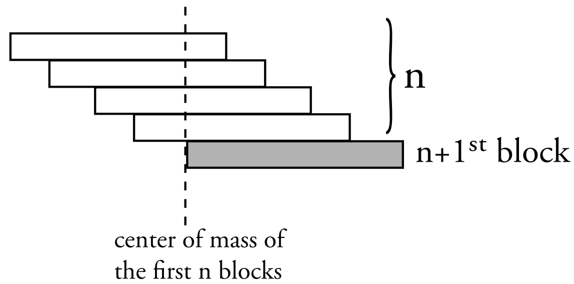

Figure 6: Stack of $n + 1$ Blocks.

So yes, you can extend this stack as far (horizontally) as you want — provided that you have enough blocks. Another way of looking at this problem is to say

$$\sum_{n=1}^{N} \frac{1}{n} = S_N$$

Recall the Riemann Sum estimation from the beginning of this lecture:

$$\ln N < S_N < (\ln N) + 1$$

as $N \to \infty$, $S_N \to \infty$.

How high would this stack of blocks be if we extended it across the two lab tables here at the front of the lecture hall? The blocks are 30 cm by 3 cm (see Figure 7). One lab table is 6.5 blocks, or 13 units, long. Two tables are 26 units long. There will be $26 - 2 = 24$ units of overhang in the stack.

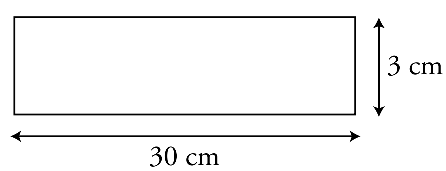

Figure 7: Side view of one block.

If $\ln N = 24$, then $N = e^{24}$.

$$\text{Height} = 3\,\text{cm} \cdot e^{24} \approx 8 \times 10^8\,\text{m}$$

That height is roughly twice the distance to the moon.

If you want the stack to span this room ($\sim$ 30 ft.), it would have to be 10^{26} meters high. That's about the diameter of the observable universe.