

Lecture 22. November 4, 2005

Homework. Problem Set 6 Part I: (f)–(h); Part II: Problem 2 (a) and (c).

Practice Problems. Course Reader: 4G-1, 4G-4, 4G-6, 4H-1, 4H-3.

1. Surface area of a right circular cone. Before attacking the general problem of the surface area of a surface of revolution, consider the simplest case of the area of a right circular cone of base radius R and height H . The *slant height* of the cone is the length of any line segment from the vertex to a point on the base circle. By the Pythagorean theorem, the slant height S is,

$$S = \sqrt{R^2 + H^2}.$$

Imagine the cone is made of paper. Make an incision along a line segment from the vertex to the base circle. The resulting piece of paper may be unfolded to form a sector of a circle. The radius of the sector is the slant height s . The circumference of the sector is the circumference of the original base circle $2\pi r$. The formula for the area and circumference of a sector of a circle give the identity,

$$\text{Area of sector} = \frac{1}{2}(\text{Radius of sector}) \times (\text{Circumference of sector}).$$

Thus, the area of the cone equals,

$$A = \frac{1}{2}(S)(2\pi R) = \pi RS.$$

In particular, the height H is involved only indirectly (as H depends on H).

Next, consider a *conical band* obtained from a right circular cone of base radius R_1 and slant height S_1 by removing the top part of the cone of base radius R_2 and slant height S_2 . In particular, the slant height of the band is the difference,

$$s = S_2 - S_1,$$

and the *average radius* of the band is the average of R_1 and R_2 ,

$$r = \frac{1}{2}(R_1 + R_2).$$

By similar triangles,

$$\frac{S_1}{R_1} = \frac{S_2}{R_2}.$$

Rearranging gives,

$$R_2 S_1 = R_1 S_2.$$

Using the formula above, the area of the large cone is,

$$A_1 = \pi R_1 S_1,$$

and the area of the small cone is,

$$A_2 = \pi R_2 S_2.$$

The area A of the band is the difference,

$$A = A_1 - A_2 = \pi(R_1 S_1 - R_2 S_2).$$

Since $R_2 S_1$ equals $R_1 S_2$, the formula is unchanged by adding $\pi R_2 S_1$ and subtracting $\pi R_1 S_2$ to get,

$$A = \pi(R_1 S_1 - R_2 S_2) + \pi(R_2 S_1 - R_1 S_2) = \pi((R_1 + R_2)S_1 - (R_1 + R_2)S_2).$$

Simplifying and substituting $R_1 + R_2 = 2r$ and $S_1 - S_2 = 2$ gives,

$$A = 2\pi r s.$$

2. Surface area of a surface of revolution. Given a segment of a parametric curve,

$$\begin{cases} x = x(t), \\ y = y(t) \end{cases} \quad a \leq t \leq b$$

the *surface of revolution* is the surface obtained by revolving the segment through xyz -space about the y -axis. What is the area of this surface? The answer is called the *surface area*.

The method for computing the surface area is so close to the method for computing the arc length of the curve, the details will be skipped. What is relevant is the differential element of surface area. Given a small interval from t to $t + dt$, approximate the segment of the parametric curve as a line segment. The surface obtained by revolving a line segment is precisely a band of a cone. The average radius of the cone r is $x(t)$. The slant height of the cone is ds . Thus the area of the band is,

$$dA = 2\pi r ds = 2\pi x(t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt.$$

Integrating gives the formula for the surface area of the surface of revolution,

$$A = \int dA = \int_{t=a}^{t=b} 2\pi x(t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt.$$

Examples. A. Consider the line segment connecting the point $(0, H)$ to the point $(R, 0)$. This has equation,

$$y = \frac{H}{R}(R - x), \quad 0 \leq x \leq R.$$

The slant height of the line segment is,

$$S = \sqrt{R^2 + H^2},$$

and the differential arc length of the line segment is,

$$ds = \frac{S}{R} dx.$$

Thus the differential element of surface area is,

$$dA = 2\pi r ds = 2\pi x \frac{S}{R} dx.$$

Integrating gives,

$$A = \int dA = \int_{x=0}^{x=R} \frac{2\pi S}{R} x dx = \frac{2\pi S}{R} \left(\frac{x^2}{2} \Big|_0^R \right) = \pi RS.$$

This is the same formula obtained above by more elementary means.

B. Consider the parametrized semicircle of radius R in the first and third quadrants,

$$\begin{cases} x = R \cos(\theta), & -\pi \leq \theta \leq \frac{\pi}{2}. \\ y = R \sin(\theta) \end{cases}$$

Revolving about the y -axis gives the *sphere* of radius R . Thus the surface area of the surface of revolution is the surface area of the sphere of radius R .

As computed in the previous lecture, the differential element of arc length is,

$$ds = R d\theta.$$

Thus the differential element of surface area is,

$$dA = 2\pi r ds = 2\pi x(\theta)(R d\theta) = 2\pi(R \cos(\theta))(R d\theta) = 2\pi R^2 \cos(\theta) d\theta.$$

Integrating gives,

$$A = \int dA = \int_{\theta=-\pi/2}^{\theta=\pi/2} 2\pi R^2 \cos(\theta) d\theta = 2\pi R^2 (\sin(\theta)) \Big|_{-\pi/2}^{\pi/2}.$$

This evaluates to,

$$A = 2\pi R^2(2) = 4\pi R^2.$$

The fastest way to remember this is to observe the surface area A and the volume V of a sphere of radius R are related by,

$$A = 4\pi R^2 = \frac{dV}{dr} = \frac{d}{dr}(4\pi R^3/3).$$

C. An *astroid* is a curve,

$$x^{2/3} + y^{2/3} = a^{2/3}.$$

The part of the astroid in the first quadrant has parametric equation,

$$\begin{cases} x = a \cos^3(t), \\ y = a \sin^3(t) \end{cases} \quad 0 \leq t \leq \frac{\pi}{2}.$$

The derivatives are,

$$\frac{dx}{dt} = -3a \cos^2(t) \sin(t), \quad \frac{dy}{dt} = 3a \sin^2(t) \cos(t).$$

Thus,

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = (-3a \cos^2(t) \sin(t))^2 + (3a \sin^2(t) \cos(t))^2 = 9a^2 \sin^2(t) \cos^2(t) (\cos^2(t) + \sin^2(t)).$$

The square root is,

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{9a^2 \sin^2(t) \cos^2(t)} = 3a \sin(t) \cos(t).$$

So the differential element of arc length is,

$$ds = 3a \sin(t) \cos(t) dt.$$

Thus the differential element of surface area of the surface of revolution is,

$$dA = 2\pi r ds = 2\pi x(t) ds = 2\pi (a \cos^3(t)) (3a \sin(t) \cos(t)) dt = 6\pi a^2 \cos^4(t) \sin(t) dt.$$

Integrating, the surface area is,

$$A = \int dA = \int_{t=0}^{t=\pi/2} 6\pi a^2 \cos^4(t) \sin(t) dt.$$

Substitute $u = \cos(t)$, $du = -\sin(t) dt$, $u(0) = 1$, $u(\pi/2) = 0$ to get,

$$A = \int_{u=1}^{u=0} 6\pi a^2 u^4 (-du) = 6\pi a^2 \int_{u=0}^{u=1} u^4 du.$$

Thus the surface area of the surface of revolution is,

$$A = 6\pi a^2 (u^5/5)|_0^1 = 6\pi a^2/5.$$

3. Polar coordinate curves. After the explicit, Cartesian form of a curve as a graph, $y = f(x)$, the next most common representation is using polar coordinates. Given a function $r = r(\theta)$ and an interval $a \leq \theta \leq b$, the associated *polar coordinate curve* is the parametric curve,

$$\begin{cases} x = r(\theta) \cos(\theta), \\ y = r(\theta) \sin(\theta) \end{cases} \quad a \leq \theta \leq b.$$

For each point on the curve, the distance of the point from the origin is,

$$\text{Distance from origin} = \sqrt{x^2 + y^2} = \sqrt{r^2} = |r| = \begin{cases} +r, & r \geq 0, \\ -r, & r < 0 \end{cases}$$

Also, assuming the point does not equal the origin, the angle of the ray from the origin to the point is,

$$\text{Angle} = \tan^{-1}(y/x) = \tan^{-1}(\tan(\theta)) = \begin{cases} \theta, & r > 0, \\ \theta + \pi, & r < 0 \end{cases}$$

This is one of the most confusing aspects of polar curves. The symbols $r(\theta)$ and θ are engrained in mathematical thinking as the distance and angle of a point in polar coordinates. But for a polar coordinate curve these are *simply parameters*. They are very closely related to, but often different from, the actual distance and angle. This is easiest to think about by imagining the point swerving through the origin along the radius line to the opposite ray of the ray given by θ . In other words, the point “goes negative”.

Given a polar curve, it is often possible to find an implicit Cartesian curve containing the polar curve.

Examples. A. Let a be a positive constant and consider the polar curve,

$$r(\theta) = a.$$

This gives,

$$r^2 = a^2 \Leftrightarrow x^2 + y^2 = a^2.$$

Thus the polar curve is contained in the circle of radius a .

B. Consider the polar curve,

$$r(\theta) = \frac{a}{\sin(\theta)}.$$

Multiplying both sides by $\sin(\theta)$ gives,

$$r \sin(\theta) = a \Leftrightarrow y = a.$$

Thus the polar curve is contained in the horizontal line passing through $(0, a)$.

C. Consider the polar curve,

$$r = 2a \cos(\theta),$$

Multiplying both sides by r gives,

$$r^2 = 2ar \cos(\theta) \Leftrightarrow x^2 + y^2 = 2ax.$$

Simplifying this gives the equation,

$$(x - a)^2 + y^2 = a^2.$$

This is the equation of the circle of radius a centered at $(a, 0)$.

4. Sketching polar curves. Given a polar curve, how are we to sketch it? For definiteness, consider the polar curve,

$$r(\theta) = \cos(2\theta), \quad -\pi/4 \leq \theta \leq 7\pi/4.$$

This curve is called the *four-leaved rose*. A similar curve occurs in Part II, Problem 2 of [Problem Set 6](#).

1. Find the range of θ . In almost every case, this will be given. In this case, the range is given as $-\pi/4 \leq \theta \leq 7\pi/4$. In some cases, the range must be determined. For instance, to sketch only the “leaf” of the rose containing $(1, 0)$, first the endpoints of this leaf must be found.

2. Determine when r is positive, zero or negative. This is easiest to keep track of with a table.

θ	r
$-\pi/4$	0
$-\pi/4 < \theta < \pi/4$	$r > 0$
$\pi/4$	0
$\pi/4 < \theta < 3\pi/4$	$r < 0$
$3\pi/4$	0
$3\pi/4 < \theta < 5\pi/4$	$r > 0$
$5\pi/4$	0
$5\pi/4 < \theta < 7\pi/4$	$r < 0$

The curve crosses the origin when $\theta = -\pi/4, \pi/4, 3\pi/4,$ and $5\pi/4$. The curve “goes negative” when $\pi/4 < \theta < 3\pi/4$ and when $5\pi/4 < \theta < 7\pi/4$.

3. Find the extremal values of $|r|$. A local maximum of $|r|$ is either a point where r is positive and a local maximum or a point where r is negative and a local minimum. Similarly for local minima of $|r|$. Typically, local maxima of $|r|$ occur either at endpoints of the interval or points where $r'(\theta)$ is zero (occasionally at discontinuity points, or nondifferentiable points). Local minima occur at such points, but also occur everytime the curve crosses the origin (so that $|r|$ equals 0).

In our example, the local minima are all points where $r = 0$, enumerated above. The derivative of r is,

$$r'(\theta) = -2 \sin(2\theta).$$

The critical points are $\theta = 0, \pi/2, \pi$ and $3\pi/2$. For $\theta = 0$ and $\theta = \pi$, r is positive and maximum. For $\theta = \pi/2$ and $\theta = 3\pi/2$, r is negative and minimum. Thus each critical point is a local maximum of $|r|$. The value of $|r|$ at each critical point is 1.

4. Find the asymptotes. This is a bit difficult with a polar curve. What is easier is to find a line parallel to an asymptote. Whenever,

$$\lim_{\theta \rightarrow a} r(\theta) = +\infty,$$

(or the same with a right-hand limit or left-hand limit), there is an asymptote parallel to the ray $\theta = a$. Whenever,

$$\lim_{\theta \rightarrow a} r(\theta) = -\infty,$$

there is an asymptote parallel to the ray $\theta = a + \pi$.

In our example, $r(\theta)$ never limits to $\pm\infty$. Thus there are no asymptotes. But in Example B., $r = a/\sin(\theta)$, r tends to $+\infty$ as θ tends to 0 from above and as θ tends to π from below. Thus there is an asymptote parallel to the x -axis. Since the explicit equation is $y = a$, which is a line parallel to the x -axis, this is correct.

5. Find the tangent direction at important points. This will be discussed further next time. The most important tangent directions are when the curve crosses the origin, and critical points of r . If $r(\theta) = 0$ and $r'(\theta) \neq 0$, the tangent line of the curve has angle θ . If $r'(\theta) = 0$ and $r(\theta) \neq 0$, the tangent line has angle $\theta \pm \pi/2$, i.e., the tangent line is orthogonal to the radius through the point. Both of these are consequences of a more general formula. The angle ψ between the tangent line and the radius satisfies,

$$\tan(\psi) = \frac{r(\theta)}{r'(\theta)}.$$

In the example, r' is nonzero whenever r is zero. Thus the tangent direction of the curve as it crosses the origin is just the direction of the limiting radius.

This is now ample information to sketch the four-leaved rose. Up to a rotation of $\pi/4$, the sketch is the same as in Figure 16.11 on p. 566 of the textbook (the sketch was also given in lecture).