

# Matrix and Index Notation

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A *vector* can be described by listing its components along the *xyz* cartesian axes; for instance the displacement vector  $\mathbf{u}$  can be denoted as  $u_x, u_y, u_z$ , using letter subscripts to indicate the individual components. The subscripts can employ numerical indices as well, with 1, 2, and 3 indicating the *x*, *y*, and *z* directions; the displacement vector can therefore be written equivalently as  $u_1, u_2, u_3$ .

A common and useful shorthand is simply to write the displacement vector as  $u_i$ , where the *i* subscript is an *index* that is assumed to range over 1,2,3 ( or simply 1 and 2 if the problem is a two-dimensional one). This is called the *range convention* for index notation. Using the range convention, the vector equation  $u_i = a$  implies three separate scalar equations:

$$u_1 = a$$

$$u_2 = a$$

$$u_3 = a$$

We will often find it convenient to denote a vector by listing its components in a vertical list enclosed in braces, and this form will help us keep track of matrix-vector multiplications a bit more easily. We therefore have the following equivalent forms of vector notation:

$$\mathbf{u} = u_i = \left\{ \begin{array}{c} u_1 \\ u_2 \\ u_3 \end{array} \right\} = \left\{ \begin{array}{c} u_x \\ u_y \\ u_z \end{array} \right\}$$

Second-rank quantities such as stress, strain, moment of inertia, and curvature can be denoted as  $3 \times 3$  matrix arrays; for instance the stress can be written using numerical indices as

$$[\sigma] = \left[ \begin{array}{ccc} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{array} \right]$$

Here the first subscript index denotes the row and the second the column. The indices also have a physical meaning, for instance  $\sigma_{23}$  indicates the stress on the 2 face (the plane whose normal is in the 2, or *y*, direction) and acting in the 3, or *z*, direction. To help distinguish them, we'll use brackets for second-rank tensors and braces for vectors.

Using the range convention for index notation, the stress can also be written as  $\sigma_{ij}$ , where both the *i* and the *j* range from 1 to 3; this gives the nine components listed explicitly above.

(Since the stress matrix is symmetric, i.e.  $\sigma_{ij} = \sigma_{ji}$ , only six of these nine components are independent.)

A subscript that is repeated in a given term is understood to imply summation over the range of the repeated subscript; this is the *summation convention* for index notation. For instance, to indicate the sum of the diagonal elements of the stress matrix we can write:

$$\sigma_{kk} = \sum_{k=1}^3 \sigma_{kk} = \sigma_{11} + \sigma_{22} + \sigma_{33}$$

The multiplication rule for matrices can be stated formally by taking  $\mathbf{A} = (a_{ij})$  to be an  $(M \times N)$  matrix and  $\mathbf{B} = (b_{ij})$  to be an  $(R \times P)$  matrix. The matrix product  $\mathbf{AB}$  is defined only when  $R = N$ , and is the  $(M \times P)$  matrix  $\mathbf{C} = (c_{ij})$  given by

$$c_{ij} = \sum_{k=1}^N a_{ik}b_{kj} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{iN}b_{Nj}$$

Using the summation convention, this can be written simply

$$c_{ij} = a_{ik}b_{kj}$$

where the summation is understood to be over the repeated index  $k$ . In the case of a  $3 \times 3$  matrix multiplying a  $3 \times 1$  column vector we have

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{Bmatrix} b_1 \\ b_2 \\ b_3 \end{Bmatrix} = \begin{Bmatrix} a_{11}b_1 + a_{12}b_2 + a_{13}b_3 \\ a_{21}b_1 + a_{22}b_2 + a_{23}b_3 \\ a_{31}b_1 + a_{32}b_2 + a_{33}b_3 \end{Bmatrix} = a_{ij}b_j$$

The *comma convention* uses a subscript comma to imply differentiation with respect to the variable following, so  $f_{,2} = \partial f / \partial y$  and  $u_{i,j} = \partial u_i / \partial x_j$ . For instance, the expression  $\sigma_{ij,j} = 0$  uses all of the three previously defined index conventions: range on  $i$ , sum on  $j$ , and differentiate:

$$\begin{aligned} \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} &= 0 \\ \frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} &= 0 \\ \frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{zy}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} &= 0 \end{aligned}$$

The *Kronecker delta* is a useful entity is defined as

$$\delta_{ij} = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$$

This is the index form of the *unit matrix*  $\mathbf{I}$ :

$$\delta_{ij} = \mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

So, for instance

$$\sigma_{kk}\delta_{ij} = \begin{bmatrix} \sigma_{kk} & 0 & 0 \\ 0 & \sigma_{kk} & 0 \\ 0 & 0 & \sigma_{kk} \end{bmatrix}$$

where  $\sigma_{kk} = \sigma_{11} + \sigma_{22} + \sigma_{33}$ .