Elastic-Plastic Fracture Mechanics

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Elastic Plastic Fracture

Previously, we have analyzed problems in which the plastic zone was small compared to the specimen dimensions (small scale yielding). In today's lecture we present techniques for analyzing situations in which there can be large scale yielding, and determine expressions for the stress components **inside** the plastic zone. We will begin with a discussion of the J integral.

J Integral

Derivation

The J integral is a line integral (path-independent) around the crack tip. It has enormous significance in elastic-plastic fracture mechanics. Key Reference: J. R. Rice, *Journal of Applied Mechanics*, 1968.

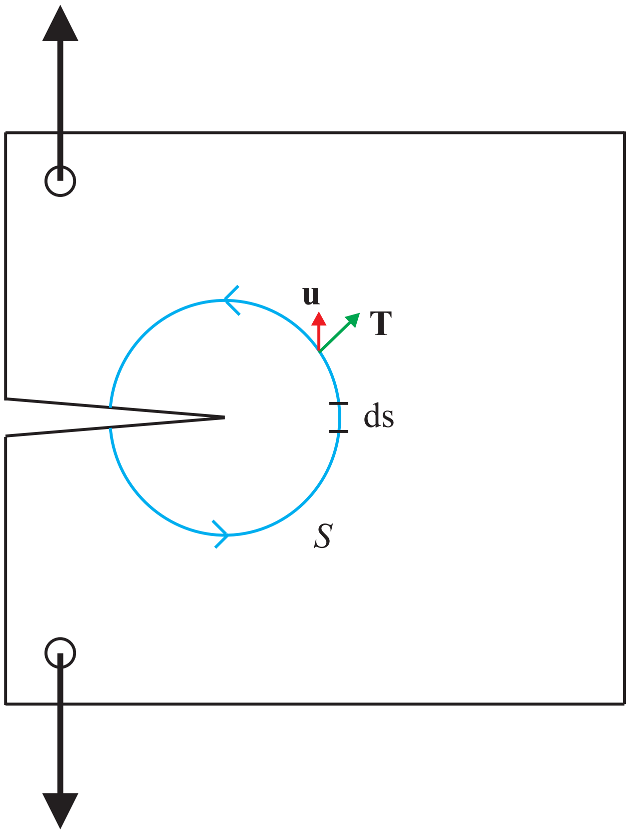
(Related works: Eshelby, *Progress in Solid State Physics* 1956; Sanders, *Journal of Applied Mechanics*, 1960; Cherepanov, *International Journal of Solids and Structures*, 1969)

J Integral

Derivation

Continued

Consider the path around the crack tip shown below:



J Integral

Derivation

Continued

We will use the following variables:

a = crack length.

S = a curve linking the lower and upper crack surfaces.

$\mathrm{d}s$ = an element of arc on this curve.

$\mathbf{T}$ = traction vector on this curve defined in relation to an outward normal unit vector, i.e. $\mathbf{T} = \mathbf{n} \cdot \boldsymbol{\sigma}$.

$\mathbf{u}$ = corresponding displacement vector.

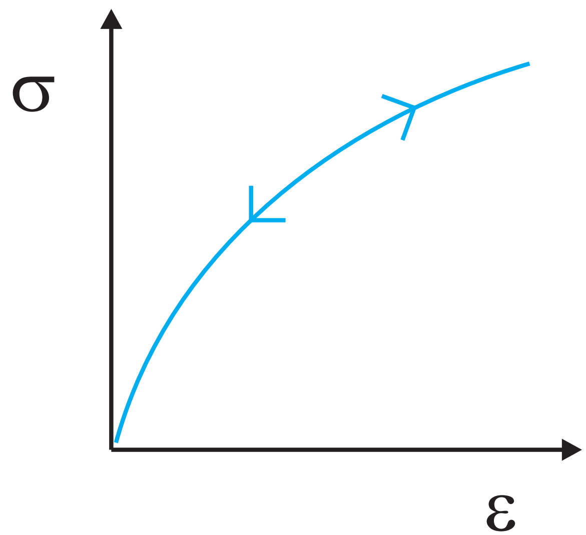
We consider a small strain analysis; we neglect any deformation-induced blunting of crack tip.

J Integral

Derivation

Continued

We use the J_2 deformation theory of plasticity (equivalent to non-linear elasticity). The (reversible) stress-strain response is depicted schematically below:



J Integral

Derivation

Continued

For **proportional** loading $\boldsymbol{J_2}$ deformation theory and $\boldsymbol{J_2}$ flow theory (incremental theory of plasticity) give results that are comparable (i.e. for monotonic loading, stationary cracks).

Not appropriate for situations where significant **unloading** occurs.

The total mechanical potential energy of the cracked body is

$$\boldsymbol{u}_{\mathrm{M}} = \boldsymbol{u}_{\mathrm{e}} + \boldsymbol{u}_{\mathrm{app}}$$

This represents the sum of the stored strain potential energy and the potential energy of the applied loading.

$\boldsymbol{J}$ Integral

Derivation

Continued

$$\boldsymbol{u}_{\mathrm{M}} = \int_{\boldsymbol{A}} \boldsymbol{w}\,\mathrm{d}\boldsymbol{A} - \int_{\boldsymbol{S}} \mathbf{T}\cdot\mathbf{u}\,\mathrm{d}\boldsymbol{s}$$

In the previous integral:

$\boldsymbol{w} =$ strain energy density (per unit volume); recall that

$$\boldsymbol{\sigma_{ij}} = \frac{\boldsymbol{\partial w}}{\boldsymbol{\partial \epsilon_{ij}}}.$$

$\mathrm{d}\boldsymbol{A}$ an element of cross section $\boldsymbol{A}$ within $\boldsymbol{S}$.

We now evaluate the derivative of the mechanical potential energy, $\boldsymbol{u}_{\mathrm{M}}$, with respect to crack length.

J Integral

Derivation

Continued

$$-\frac{\mathrm{d}\boldsymbol{u}_{\mathrm{M}}}{\mathrm{d}\boldsymbol{a}} = \int_{\boldsymbol{s}} \left(\boldsymbol{w}\mathrm{d}\boldsymbol{y} - \mathbf{T} \cdot \frac{\boldsymbol{\partial}\mathbf{u}}{\boldsymbol{\partial x}} \mathrm{d}\boldsymbol{s} \right)$$

$$\equiv \boldsymbol{J}$$

$\boldsymbol{J}$ represents the rate of change of net potential energy with respect to crack advance (per unit thickness of crack front) for a non-linear elastic solid. $\boldsymbol{J}$ also can be thought of as the energy flow into the crack tip. Thus, $\boldsymbol{J}$ is a measure of the singularity strength at the crack tip for the case of elastic-plastic material response.

J Integral

Derivation

Continued

For the special case of a **linear elastic** solid,

$$\boldsymbol{J} = \boldsymbol{\mathcal{G}} = -\frac{\mathrm{d}\,(\mathsf{PE})}{\mathrm{d}\boldsymbol{a}} = -\frac{\mathrm{d}\boldsymbol{U_M}}{\mathrm{d}\boldsymbol{a}}$$

$$= \frac{\boldsymbol{K^2}}{\boldsymbol{E}}\left(\mathbf{1} - \boldsymbol{\nu}^{\mathbf{2}}\right)$$

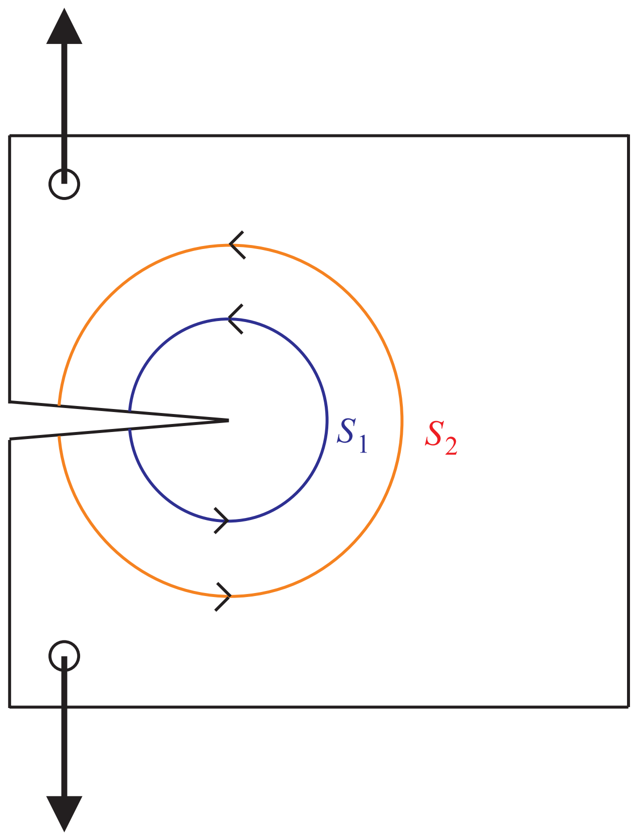
This relationship can be used to infer an equivalent $\boldsymbol{K_{Ic}}$ value from $\boldsymbol{J_{Ic}}$ measurements in high toughness, ductile solids in which valid $\boldsymbol{K_{Ic}}$ testing will require unreasonably large test specimens.

J Integral

Derivation

Continued

Consider two different paths around the crack tip:



J Integral

Derivation

Continued

$$J_{\text{along } s_1} = J_{\text{along } s_2}$$

The J Integral is independent of the path around the crack tip.

If S_2 is in elastic material,

$$J_{S_2} = \frac{K^2}{E}\left(1 - \nu^2\right)$$

HRR field

We now consider the Hutchinson, Rice, Rosengren (HRR) singular crack tip fields for **elastoplastic** material response. (Recall Williams solution assumes *linear elastic* material behavior).

Assume: Pure power law material response:

$$\frac{\epsilon}{\epsilon_0} = \alpha \left(\frac{\sigma}{\sigma_0} \right)^n$$

α = material constant, σ_0 = reference yield strength, n = strain hardening exponent, $\epsilon_0 =$ reference yield strain $= \sigma_0 / E$.

For linear elastic material $n = 1$, for perfectly plastic response $n = \infty$.

HRR field

Continued

With these assumptions, the crack tip fields (HRR field) can be derived. (Ref: J.W. Hutchinson, *JMPS*, 1968 and J.R. Rice and G.F. Rosengren, *JMPS*, 1968.)

$$\sigma_{ij} = \sigma_0 \left(\frac{J}{\alpha \sigma_0 \epsilon_0 I_n r} \right)^{\frac{1}{n+1}} \tilde{\sigma}_{ij}\left(\theta, n\right)$$

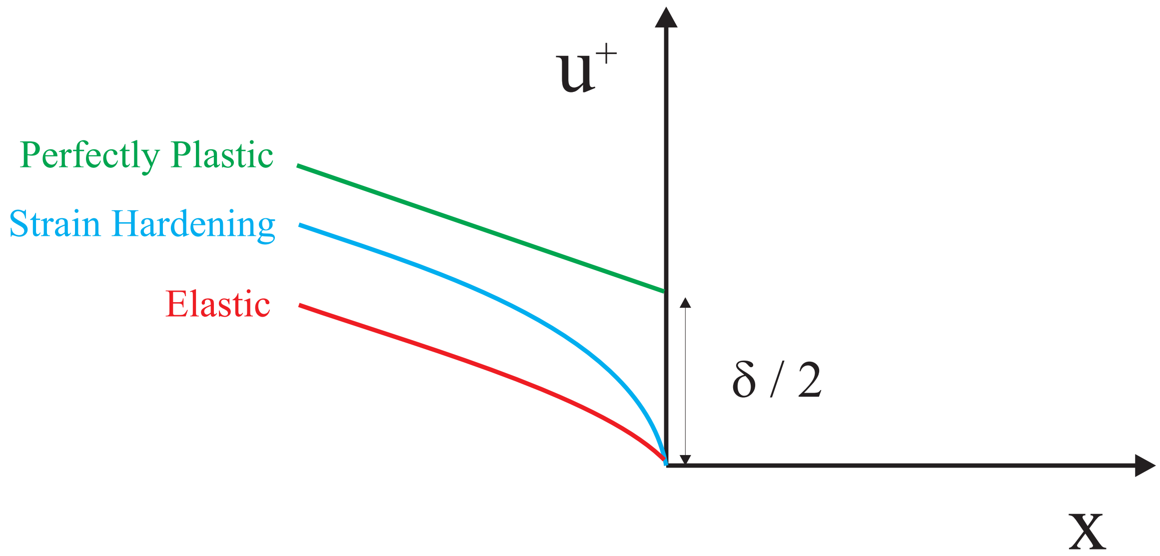
$$\epsilon_{ij} = \alpha \epsilon_0 \left(\frac{J}{\alpha \sigma_0 \epsilon_0 I_n r} \right)^{\frac{n}{n+1}} \tilde{\epsilon}_{ij}\left(\theta, n\right)$$

$$u_i = \alpha \epsilon_0 \left(\frac{J}{\alpha \sigma_0 \epsilon_0 I_n r} \right)^{\frac{n}{n+1}} r^{\frac{1}{n+1}} \tilde{u}_i\left(\theta, n\right)$$

The function I_n has a weak dependence on n.

CTOD

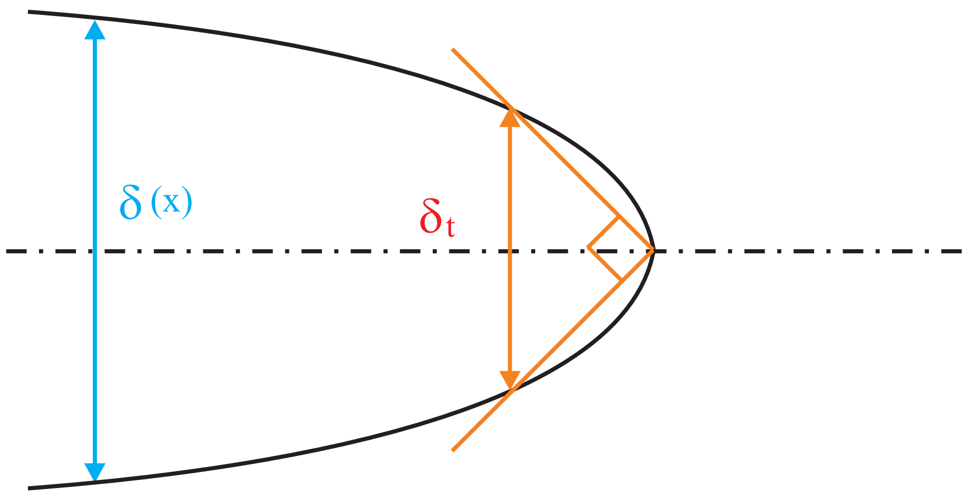
The variation in crack tip opening displacement δ_t or (CTOD) for different material response is depicted below:



The crack tip opening displacement depends on distance from the crack tip. We need an operational definition for CTOD.

CTOD

The definition of δ_t is somewhat arbitrary since the opening displacement varies as the crack tip is approached. A commonly used operational definition is based on the $45°$ construction depicted below (see C.F. Shih, *JMPS*, 1982).



CTOD

$$\delta_t = d_n \frac{J}{\sigma_0}$$

d_n is a strong function of n, and a weak function of σ_0/E.

Plane Strain:

$$d_n \approx 0.3 - 0.65 \ (0.65 \text{ for n} \rightarrow \infty)$$

Plane Stress:

$$d_n \approx 0.5 - 1.07 \ (1.07 \text{ for n} \rightarrow \infty)$$

CTOD

Presuming dominance of HRR fields

$$\delta_t = d_n \frac{J}{\sigma_0} \approx \frac{J}{\sigma_0}$$

For Small Scale Yielding (SSY)

$$J = \frac{K_I^2}{E}\left(1 - \nu^2\right)$$

$$\delta_t = d_n \frac{K_I^2}{E} \frac{\left(1 - \nu^2\right)}{\sigma_0}$$

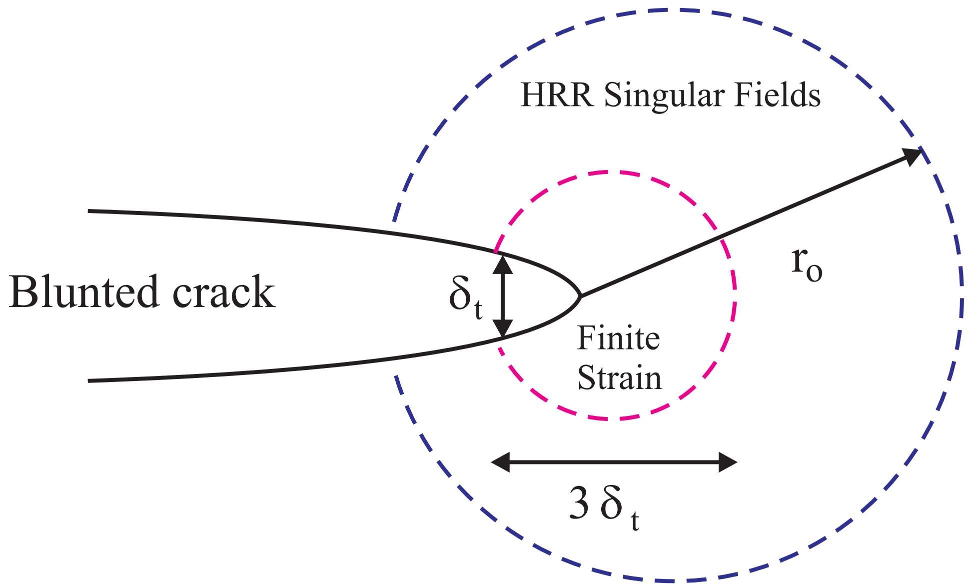
CTOD

Importance/Applications of CTOD:

- Critical CTOD as a measure of toughness.
- Exp. measure of driving force.
- Multiaxial fracture characterization.
- Specimen size requirements for $\boldsymbol{K_{Ic}}$ and $\boldsymbol{J_{Ic}}$ testing.

J-Dominance

Just as for the $\boldsymbol{K}$ field, there is a domain of validity for the HRR ($\boldsymbol{J}$-based) fields.



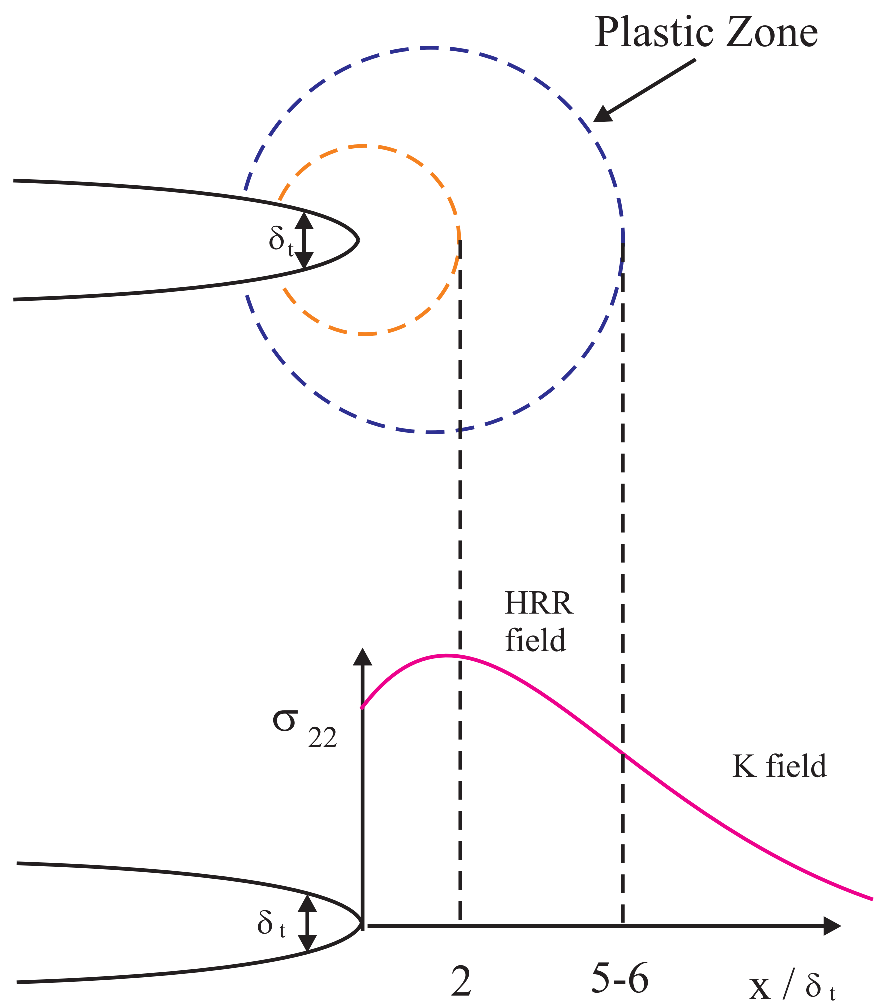
J-Dominance

Under plane strain and small scale yielding conditions, it has been found that:

$$ {}_0 \approx \frac{1}{4} r_p $$

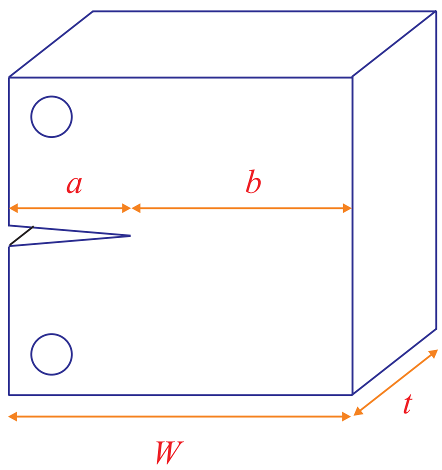
For J dominance the uncracked ligament size b must be greater than 25 times the CTOD or $\approx 25 \times J/\sigma_0$. The variation in stress ahead of the crack is depicted on the following page:

J-Dominance



K and J-Dominance

Consider a low strength steel with $\boldsymbol{\sigma_0 = 350}$ MPa, $\boldsymbol{K_{Ic} = 250}$ MPa$\sqrt{\text{m}}$ and $\boldsymbol{E = 210}$ GPa. What are the **Minimum** specimen size requirements for valid $\boldsymbol{K_{Ic}}$ and $\boldsymbol{J_{Ic}}$ measurements?



K-Dominance

ASTM standard E399 (1974) for K_{Ic} testing:

$$a, b, t > 2.5 \left(\frac{K_{Ic}}{\sigma_0} \right)^2$$

Substitute the known values for σ_0 and K_{Ic}. Find that

$$a, b, t > 1.28 \text{ m! } (\approx 50 \text{ inches})$$

J-Dominance

For J_{Ic} testing, the condition requires that for a deeply cracked compact tension or bend specimen:

$$b > 25\frac{J_{Ic}}{\sigma_0} = 25\frac{K_{Ic}}{E}\frac{\left(1-\nu^2\right)}{\sigma_0}$$

$$b > 0.02 \text{ m}$$

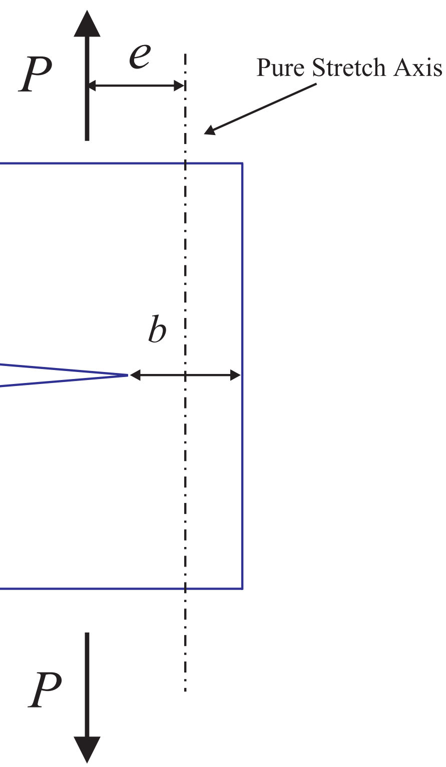
Specimen size requirements for J testing are much less severe than for K testing.

J-Dominance

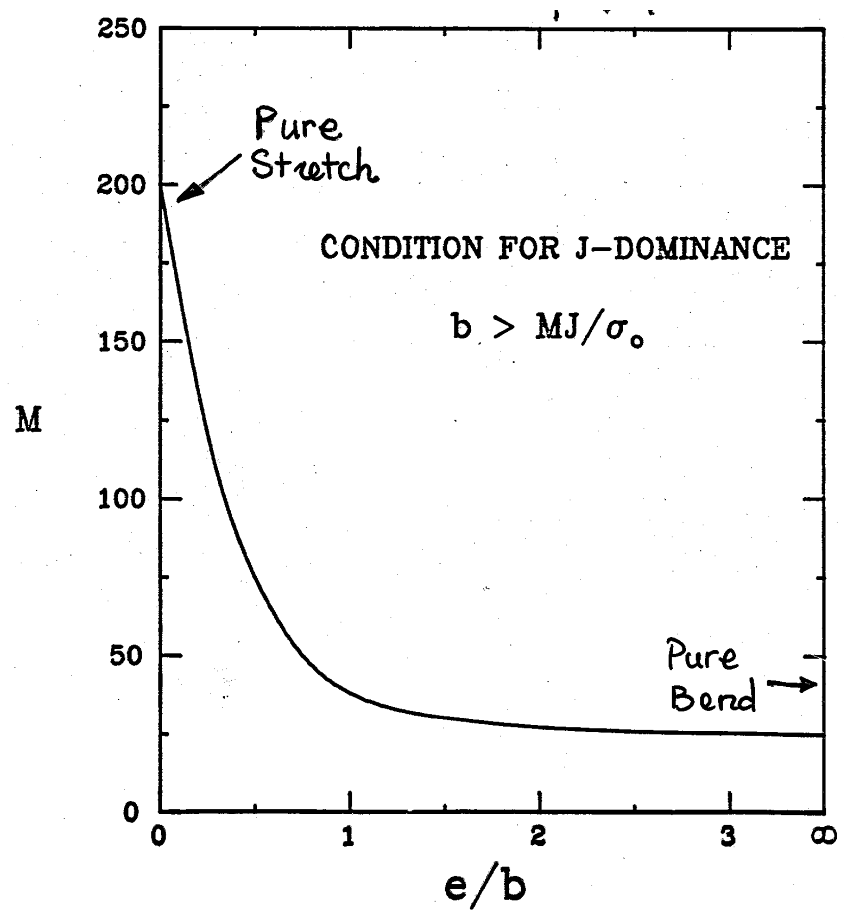
The measured J_{Ic} value may be converted to equivalent K_{Ic} value. The validity of this approach has been verified by extensive testing.

J-Dominance

Example: notched bar loaded axially (induces bending and stretching)



J-Dominance



Pure Stretch
CONDITION FOR J–DOMINANCE
b > MJ/σ_o
Pure Bend
M
e/b
250
200
150
100
50
0
0
1
2
3
∞