

3.225: Electrical and Mechanical Properties of Materials  
Test 1  
October 25, 2005

1.

A fiber composite is transversely isotropic with the following elastic moduli:

The composite is loaded by the following stress state:

$$E_1 = E_2 = 10GPa$$

$$E_3 = 40GPa$$

$$\nu_{12} = 0.2$$

$$\nu_{13} = 0.3$$

$$G_{13} = 8GPa$$

- Calculate the strain matrix.
- Calculate the elastic strain energy in the composite.

2.

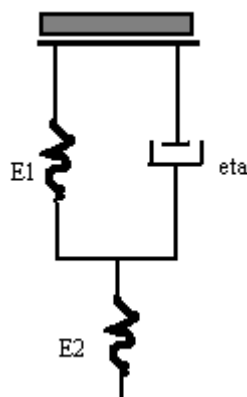
(a) Silica glass and rubbers are both amorphous. Why is the Young's modulus of the silica glass governed by the change in binding energy with strain while that of a rubber is governed by the change in entropy with strain?

(b) Why is Poisson's ratio almost 1/2 for a rubber?

3.

Derive a constitutive equation for the standard linear solid, which shows how the overall stress, stress state, strain and strain rate are related by the three model parameters:

$$E_1, E_2, \eta$$



Show your derivation clearly.

4.

An aluminum component ( $\sigma_y = 200\text{MPa}$ ) is loaded under the stress state below.

$$\sigma = \begin{bmatrix} 10 & 0 & 5 \\ 10 & 100 & 0 \\ 0 & 0 & 100 \end{bmatrix} \text{MPa}$$

- (a) Does the component yield (use the von Mises yield criterion).
- (b) What is the mean stress? The deviatoric stress?
- (c) Derive equations for the strengthening of alloys by precipitation hardening and by dispersion hardening.
- (d) Why are covalently bonded materials intrinsically hard while metals are intrinsically soft?

5.

A 100mm wide, 10mm thick glass plate has a 2mm long edge crack. The plate is loaded in tension with a stress of 10 MPa. The fracture toughness of the glass is  $1 \text{ MN m}^{-3/2}$  and the yield strength is 3600 MPa.

- (a) Does the glass plate fracture under this stress?
- (b) What is the size of the plastic zone (assume plane strain)?
- (c) Does this configuration meet the small scale yielding requirement for the use of the Griffith criterion?
- (d) If the plate was made from mild steel (fracture toughness of  $140 \text{ MNm}^{-3/2}$  and yield strength of 210 MPa) would the configuration meet the small scale yielding requirement for the use of the Griffith criterion?

## Solutions

1.

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{bmatrix} = \begin{bmatrix} 0.1 & -0.02 & -0.03 & & & \\ -0.02 & 0.1 & -0.03 & & & \\ -0.03 & -0.03 & 0.025 & & & \\ & & & 0.125 & & \\ & & & & 0.125 & \\ & & & & & ? \end{bmatrix} \times 10^{-3} \text{MPa}^{-1} \begin{bmatrix} 10 \\ 4 \\ 15 \\ 6 \\ 5 \\ 0 \end{bmatrix} \text{MPa} = \begin{bmatrix} 4.7 \times 10^{-4} \\ -2.5 \times 10^{-4} \\ -4.5 \times 10^{-5} \\ 7.5 \times 10^{-4} \\ 6.25 \times 10^{-4} \\ 0 \end{bmatrix}$$

$$\varepsilon = \begin{bmatrix} 4.7 \times 10^{-4} & 0 & 6.25 \times 10^{-4} \\ 0 & -2.5 \times 10^{-4} & 7.5 \times 10^{-4} \\ 6.25 \times 10^{-4} & 7.5 \times 10^{-4} & -4.5 \times 10^{-5} \end{bmatrix}$$

$$U = \frac{1}{2} \sum \sigma_i \varepsilon_i = 5325 \text{ J} / \text{m}^3$$

2.

a) Silica glass is amorphous material in which atoms are bonded to each surrounding atom. Under uniaxial deformation, the bonds stretch but the relative positions of the atoms do not change much, so that there is little change in entropy with straining. The Young's modulus depends on the change in the binding energy with strain. Rubbers are polymer chains with strong covalent bonds along the chains but only occasional covalent cross-links between the chains. Under uniaxial deformation, the chains can slide over one another, making the structure more ordered, and giving large changes in entropy. As the chains slide over one another, the bonds along the chains are not stretched much, so that there is little change in the binding energy.

b) Under hydrostatic loading, the polymer chains in a rubber retain their relative position, but get either closer together (in compression) or further apart (in tension), stretching the bonds. The bulk modulus of rubbers depends on the change in the binding energy with strain and is large relative to the Young's modulus (as it is much more difficult to stretch the bonds along the chain than to slide the chains over one another, as in uniaxial stress). Using the relationships between K and E for isotropic solids, it can be shown that for large K/E, Poisson's ratio approaches 1/2.

3., 4., 5.

(See handwritten solution)

$$\#1 \quad \nu_{ij} = - \frac{\epsilon_j}{\epsilon_i}$$

$$S_{12} = ?$$

apply  $\sigma_1$  only:

$$\epsilon_1 = S_{11} \sigma_1$$

$$\epsilon_2 = S_{12} \sigma_1$$

$$\nu_{12} = - \frac{\epsilon_2}{\epsilon_1} = - \frac{S_{12} \sigma_1}{S_{11} \sigma_1}$$

$$S_{12} = - \nu_{12} S_{11} = - \frac{\nu_{12}}{E_1}$$

$$\epsilon_1 = S_{11} \sigma_1$$

$$\epsilon_3 = S_{13} \sigma_1$$

$$\nu_{13} = - \frac{\epsilon_3}{\epsilon_1} = - \frac{S_{13} \sigma_1}{S_{11} \sigma_1} = - \frac{S_{13}}{S_{11}}$$

$$S_{13} = - \frac{\nu_{13}}{E_1}$$

$$\nu_{12} = 0.2 \quad E_1 = 10 \text{ GPa} \quad S_{12} = -0.2/10 = -0.02 \text{ GPa}^{-1}$$

$$\nu_{13} = 0.3 \quad E_1 = 10 \text{ GPa} \quad S_{13} = -0.3/10 = -0.03 \text{ GPa}^{-1}$$

$$\nu_{23} = \nu_{13} = -0.03 \text{ GPa}^{-1}$$

$$S_{12} = 2(S_{11} - S_{12}) = 2(0.1 - (-0.02)) = 0.240 \text{ GPa}^{-1}$$

$$S_{11} = 1/E_1 = 0.1 \text{ GPa}^{-1} \quad S_{22} = 1/E_2 = 0.1 \text{ GPa}^{-1} \quad S_{33} = 1/E_3 = 0.025 \text{ GPa}^{-1}$$

$$S_{44} = 1/6_{23} = 1/6_{10} = 0.125 \text{ GPa}^{-1} \quad S_{55} = 1/6_{13} = 0.125 \text{ GPa}^{-1}$$

$$\begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_4 \\ \epsilon_5 \\ \epsilon_6 \end{bmatrix} = \begin{bmatrix} 0.1 & -0.02 & -0.03 & & & \\ -0.02 & 0.1 & -0.03 & & & \\ -0.03 & -0.03 & 0.025 & & & \\ & & & 0.125 & & \\ & & & & 0.125 & \\ & & & & & 0.240 \end{bmatrix} \times 10^{-3} \text{ MPa} \begin{bmatrix} 10 \\ 4 \\ 15 \\ 6 \\ 5 \\ 3 \end{bmatrix} \text{ MPa} = \begin{bmatrix} 4.7 \times 10^{-4} \\ -2.15 \times 10^{-4} \\ -4.5 \times 10^{-5} \\ 7.5 \times 10^{-4} \\ 6.25 \times 10^{-4} \\ 7.20 \times 10^{-4} \end{bmatrix}$$

$$U = \frac{1}{2} \sum \sigma_i \epsilon_i = 6410.5 \text{ J/m}^3$$

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**Test 1 October 25, 2005**  
**Solutions**

2. (a) Silica glass and rubbers are both amorphous. Why is the Young's modulus of the silica glass governed by the change in binding energy with strain while that of a rubber is governed by the change in entropy with strain?

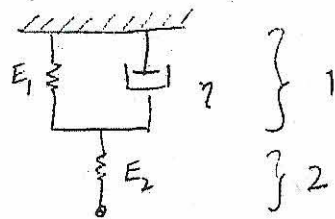
Silica glass is amorphous material in which atoms are bonded to each surrounding atom. Under uniaxial deformation, the bonds stretch but the relative positions of the atoms do not change much, so that there is little change in entropy with straining. The Young's modulus depends on the change in the binding energy with strain. Rubbers are polymer chains with strong covalent bonds along the chains but only occasional covalent cross-links between the chains. Under uniaxial deformation, the chains can slide over one another, making the structure more ordered, and giving large changes in entropy. As the chains slide over one another, the bonds along the chains are not stretched much, so that there is little change in the binding energy.

2. (b) Why is Poisson's ratio almost 1/2 for a rubber?

Under hydrostatic loading, the polymer chains in a rubber retain their relative position, but get either closer together (in compression) or further apart (in tension), stretching the bonds. The bulk modulus of rubbers depends on the change in the binding energy with strain and is large relative to the Young's modulus (as it is much more difficult to stretch the bonds along the chain than to slide the chains over one another, as in uniaxial stress). Using the relationships between  $K$  and  $E$  for isotropic solids, it can be shown that for large  $K/E$ , Poisson's ratio approaches 1/2.

4. (d) From the expression for intrinsic lattice resistance we see that  $\tau_o = \frac{4}{3} \frac{E}{z}$  where  $E$  is the Young's modulus and  $z$  is the number of bonded neighbours per atom. Since  $E$  is of the same order for both materials, the intrinsic lattice resistance varies with  $1/z$ . For covalently bonded materials,  $U_{\text{attr}} \propto \frac{1}{r^m}$ , where  $m \sim 6$ . This implies that  $U_{\text{attr}}$  decreases rapidly with increasing  $r$ , so that for covalently bonded materials only the nearest neighbour interactions are significant and they are intrinsically hard. For metals, there are free electrons and non-localized bonds, so that  $z$  is much higher, making them intrinsically soft.

3.



For the whole unit

$$\epsilon = \epsilon_1 + \epsilon_2, \quad \dot{\epsilon} = \dot{\epsilon}_1 + \dot{\epsilon}_2, \quad \sigma = \sigma_1 = \sigma_2, \quad \dot{\sigma} = \dot{\sigma}_1 = \dot{\sigma}_2$$

For the lone spring

$$\sigma_2 = E_2 \epsilon_2, \quad \dot{\sigma}_2 = E_2 \dot{\epsilon}_2$$

The Varget Element

$$\sigma_1 = \sigma_{1S} + \sigma_{1D}, \quad \dot{\sigma}_1 = \dot{\sigma}_{1S} + \dot{\sigma}_{1D}, \quad \epsilon_1 = \epsilon_{1S} = \epsilon_{1D}, \quad \dot{\epsilon}_1 = \dot{\epsilon}_{1S} = \dot{\epsilon}_{1D}$$

The Spring in the Varget Element

$$\sigma_{1S} = E_1 \epsilon_{1S}, \quad \dot{\sigma}_{1S} = E_1 \dot{\epsilon}_{1S}$$

The Dashpot:

$$\sigma_{1D} = \eta \dot{\epsilon}_{1D}$$

Simplifying

$$\sigma_{1D} = \sigma_1 - \sigma_{1S} = \sigma - \sigma_{1S} = \eta \dot{\epsilon}_{1D} = \eta \dot{\epsilon}_1$$

$$\Rightarrow \eta \dot{\epsilon}_1 = \sigma - E_1 \epsilon_1 \quad \text{or} \quad \sigma = E_1 \epsilon_1 + \eta \dot{\epsilon}_1$$

$$\dot{\epsilon}_1 = \dot{\epsilon} - \dot{\epsilon}_2 = \dot{\epsilon} - \frac{\dot{\sigma}_2}{E_2} = \dot{\epsilon} - \frac{\dot{\sigma}}{E_2}$$

$$\epsilon_1 = \epsilon - \epsilon_2 = \epsilon - \frac{\sigma}{E_2} = \epsilon - \frac{\sigma}{E_2}$$

Plugging the last 2 into the  $\sigma$  equation

$$\sigma = \eta \left( \dot{\epsilon} - \frac{\dot{\sigma}}{E_2} \right) + E_1 \left( \epsilon - \frac{\sigma}{E_2} \right)$$

Rearranging

$$\boxed{\eta \dot{\epsilon} + E_1 \epsilon = \frac{\eta}{E_2} \dot{\sigma} + \left( 1 + \frac{E_1}{E_2} \right) \sigma}$$

#4

$$\sigma = \begin{bmatrix} 50 & 10 & 0 \\ 10 & 100 & 0 \\ 0 & 0 & 100 \end{bmatrix}$$

$$\begin{aligned} (a) \quad \sigma_{eq} &= \sqrt{\frac{1}{2} \{ (\sigma_{11} - \sigma_{22})^2 + (\sigma_{22} - \sigma_{33})^2 + (\sigma_{33} - \sigma_{11})^2 \} + 3\sigma_{12}^2 + 3\sigma_{13}^2 + 3\sigma_{23}^2} \\ &= \sqrt{\frac{1}{2} \{ 50^2 + 0 + 50^2 \} + 3(10^2)} \\ &= 52.9 \text{ MPa.} \end{aligned}$$

No yield

$$(b) \quad \sigma_m = \frac{50 + 100 + 100}{3} = 83.3 \text{ MPa}$$

$$S = \begin{bmatrix} 50 - 83.3 & 10 & 0 \\ 10 & 100 - 83.3 & 0 \\ 0 & 0 & 100 - 83.3 \end{bmatrix} = \begin{bmatrix} -33.3 & 10 & 0 \\ 10 & 16.7 & 0 \\ 0 & 0 & 16.7 \end{bmatrix}$$

(c) see notes for solution

(d) in intrinsic lattice resistance depends on # neighbouring atoms contributing to the bonding.  $\tau^0 = \frac{E}{4z}$

covalently bonded materials - only nearest neighbour atoms contribute  $\approx$  small (e.g.  $z \sim 12 \Rightarrow \tau^0 = E/36$ )

metals - free electrons - many atoms contribute,  $\approx$  large

$$\tau^0/E \sim 10^{-4} \quad \text{fcc metals.}$$

$$\begin{aligned} \underline{5.} \quad w &= 100 \text{ mm} & \sigma &= 10 \text{ MPa} \\ B &= 10 \text{ mm} & K_{Ic} &= 1 \text{ MN/m}^{3/2} \\ a &= 2 \text{ mm} & \sigma_y &= 3600 \text{ MPa} \end{aligned}$$

$$\begin{aligned} (a) \quad K_I &= 1.12 \sigma \sqrt{\pi a} \\ &= 1.12 \times (10 \times 10^6) \sqrt{\pi \times 2 \times 10^{-3}} \text{ N/m}^{3/2} \\ &= 0.88 \text{ MN/m}^{3/2} \end{aligned}$$

Since  $K_I < K_{Ic} \rightarrow$  NO FRACTURE

$$\begin{aligned} (b) \quad r_p &= \frac{1}{3\pi} \left( \frac{K_I}{\sigma_{ys}} \right)^2 = \frac{1}{3\pi} \times \left( \frac{0.88}{3600} \right)^2 \\ &= \underline{6.33 \text{ nm}} \end{aligned}$$

$$(c) \quad SSY = \frac{r_p}{a} < 0.02 \quad (\text{OK})$$

$$1) \quad \text{Steel } K_{Ic} = 140 \text{ MN/m}^{3/2} \quad \sigma_{ys}' = 210 \text{ MPa}$$

$$r_p = \frac{1}{3\pi} \left( \frac{K_I}{\sigma_{ys}' } \right)^2 = \frac{1}{3\pi} \left( \frac{0.88}{210} \right)^2 = \underline{1.863 \times 10^{-6} \text{ m}}$$

$$\frac{r_p}{a} = \frac{1.863 \times 10^{-6}}{2 \times 10^{-3}} = \underline{9.315 \times 10^{-4} < 0.02}$$

Meets SSY (criterion)