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3.22 Mechanical Properties of Materials Spring 2008

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Quiz #1 Example Solutions

Spring 2008

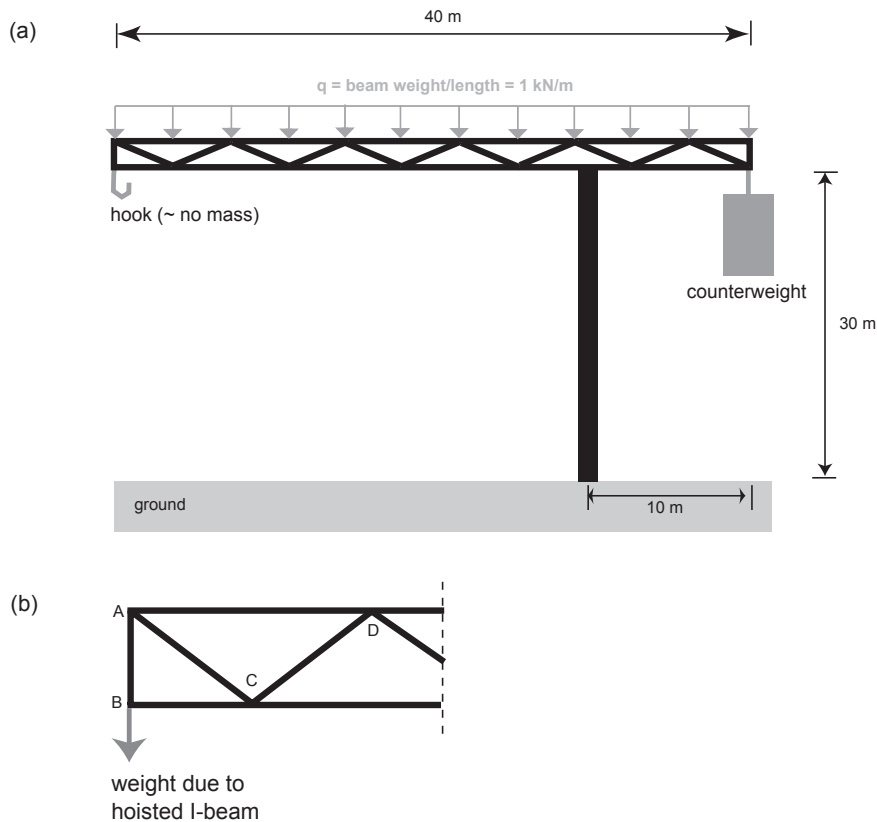
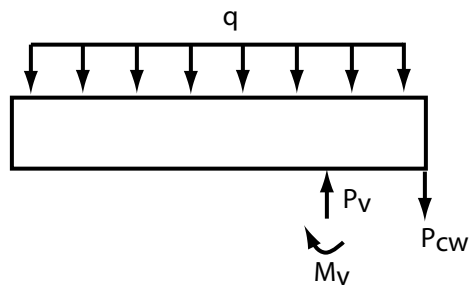


Figure 1: Schematic of crane used to complete construction along the Charles River, indicated with the horizontal element of the crane (a trussed beam) parallel to the ground. (a) Entire crane; (b) Hook-end of the horizontal element.

1. (a) Draw a free body diagram of the horizontal element of the crane (the horizontal span in Fig. 1a), treating it as a beam and indicating all known and unknown applied / reaction forces. You will need to make assumptions about how the crane is connected to the real world, and these should be *briefly* justified.

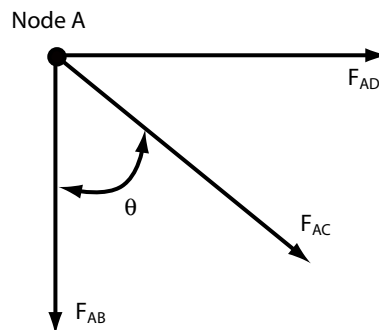
Solution: The purpose of this question is to confirm the basics of structural mechanics required to implement understanding of material mechanical behavior in context of devices. The moment about the vertical support point v , M_v equals zero if we treat it as a pin. However it does not equal zero if we treat it as a clamp. We definitely know that it is not a roller because it is not free to translate.

- (b) Consider the hook-end of the crane's horizontal element (Fig. 1b), now loaded by the weight of the steel I-beam. Can member AC be made from a thin steel cable (essentially



an elastic rope)? Here, you're considering the elements of the truss explicitly, so their weight as a whole beam structure (i.e., q) can be neglected. Explain your answer fully.

Solution: Because of the tensile applied load at the hook, segment AB is in tension. The sum of the forces in the y-direction must be balanced at node A (see figure below). Therefore, the force along segment AC is in compression to balance the tensile force on AB. Since buckling may occur when the thin steel cable is loaded in compression, it would not be advisable to make the AC member out of cable.



2. (a) We know σ_{ij} and ϵ_{ij} can be expressed as quadrics, three-dimensional surfaces defined by a cubic polynomial equation of σ_p or ϵ_p . Explain concisely, but fully and accurately, what this fact immediately implies about tensorial stress and strain.

Solution: The equation defining the quadratic has three principal roots (principal stresses or strains) along the axes defining normals to planes on which no shear components exist. This absence of shear components in the principal stress state is one of the most important concepts in stress/strain tensor transformations, and is defined explicitly by the quadric or principal root equation that includes the invariants I_1 , I_2 , and I_3 . (It is also true that the quadric form demonstrates symmetry of shear elements in the 2nd rank stress and strain tensors.)

- (b) You are asked to confirm that your thesis advisor's following calculations are correct. (He is an expert in electromagnetic material behavior, and often leans on you for help with mechanical behavior.) He has fabricated a metal interconnect line from vapor deposited Cu, and x-ray diffraction has indicated that all the grains in the polycrystalline line are oriented with all grain boundaries inclined 45° with respect to the Si substrate. When the device is heated, your advisor expects the material at the center of the line to expand uniformly in all directions (neglecting the edge effects at the substrate and free surface). He therefore predicts that this means the line will probably fail because the grains will shear apart at their boundaries. Respectfully explain to him why this prediction is incorrect, even if we allow for neglect of edge effects.

Solution: In thermal expansion, a hydrostatic strain/stress (depending on the material constraints) forms if edge effects are neglected. By definition, a hydrostatic strain/stress has no shear components. Therefore, the grains will not be under a driving force for shear failure.

Some students noted that elastic isotropy is unlikely in this case, which is true because the lines are likely textured (have the same unit cell orientation within each grain). That is probably true, though not explicitly stated. Even so, the shear component for an linear elastic / anisotropic material would be minimal under an isotropic thermal field.

- (c) He then asks you to check a stress transformation onto the grain boundary planes, calculated by your labmate, as shown below. Is this transformation correct? (You do not need to repeat the transformation to answer this.)

$$\sigma_{ij} = \begin{vmatrix} 1 & 3 & 10 \\ 4 & 5 & 7 \\ 8 & 1 & 12 \end{vmatrix} \rightarrow \sigma_{i'j'} = \begin{vmatrix} 3 & 7 & 4 \\ 1 & 11 & 8 \\ 6 & 5 & 2 \end{vmatrix}$$

Solution: No. The trace or first invariant I_1 of the original "tensor" and the transformed "tensor" (which is also asymmetric) are not equal. Also, the original stress tensor is not actually a tensor in the first place because it is asymmetric (e.g. $\sigma_{ij} \neq \sigma_{ji}$).

3. A polycrystalline, pure silver (Ag) slug is placed in a rigid die as shown, to create custom jewelry. Uniaxial compressive stress $\sigma_{zz} = \sigma_o$ is applied normal to the slug face. We will assume the die is well lubricated at the material/die interface.

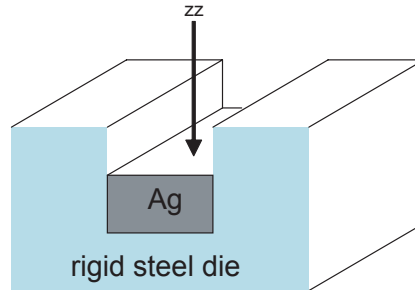


Figure 2: Ag slug compressed within rigid die.

- (a) Determine the orthogonal stresses σ_{xx} and σ_{yy} in terms of the applied stress and appropriate elastic constants of the material. Here, note that there are constrained and free surfaces created by the geometry of the die, such that the *applied stress* is uniaxial but the stress state in the material is more complex.

Solution: The applied stress σ_o is compressive. The slug is constrained in what I assigned to be the y-direction. Therefore, the strain in the y-direction ϵ_y and the stress in the x-direction σ_x are zero. We can then determine the stress in the y-direction by

$$\begin{aligned}\epsilon_y &= \frac{\sigma_y}{E} - \nu \left(\frac{\sigma_x}{E} + \frac{\sigma_z}{E} \right) \\ 0 &= \frac{\sigma_y}{E} - \nu \left(0 + \frac{\sigma_o}{E} \right) \\ \sigma_y &= -\nu\sigma_o\end{aligned}$$

Finally because the slug is unconstrained in the x-direction, $\sigma_x = 0$.

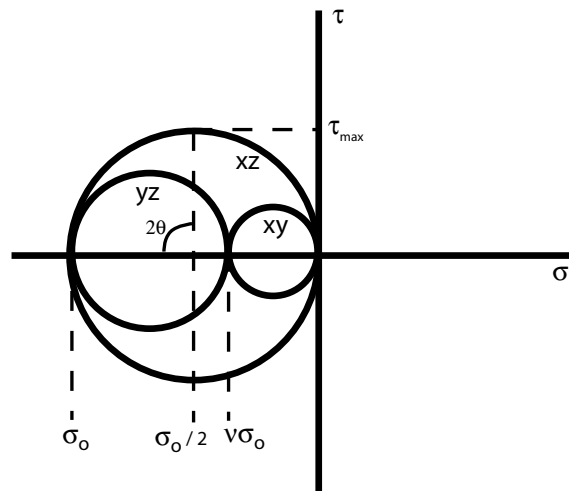
- (b) Determine the axial strain ϵ_{zz} in terms of the applied stress and appropriate elastic constants of the material.

Solution:

$$\begin{aligned}\epsilon_z &= \frac{\sigma_z}{E} - \nu \frac{\sigma_y}{E} \\ \epsilon_z &= \frac{-\sigma_o}{E} - \nu \frac{-\nu\sigma_o}{E} \\ \epsilon_z &= \frac{\sigma_o}{E} (1 - \nu^2)\end{aligned}$$

- (c) This material will deform under conditions of high shear stress. Predict the orientation and magnitude of maximum shear stress resulting from this applied stress state with respect to the given axis-set, being sure to indicate the plane (xy, xz, or yz) on which this maximum shear stress occurs.

Solution: This can be determined by constructing a Mohr's circle for each plane. The



maximum shear is on the xz-plane and equal to $\sigma_0/2$. The orientation $\theta = 45^\circ$ is given by $2\theta = 90^\circ$ and is counter-clockwise to the original axes on the xz-plane.

4. (a) When discussing the atomistic basis of linear elastic deformation, we considered an approximate derivation that related the Young's elastic modulus E to the internal energy U as a function of strain ϵ , and then a more rigorous definition in terms of $U(r)$. In contrast, for a rubber elastic / hyperelastic material, we only discussed the rigorous atomistic basis in terms of a spring constant. Briefly explain the difference between E and k in terms of units and properties of the material.

Solution: For a spring constant k , we define it as a force F per unit distance x such that

$$k = \frac{F}{x} = \frac{\text{Newtons}}{\text{meter}}$$

and this reflects interatomic bond stiffness (2nd derivative of $U(r)$ evaluated at the equilibrium interatomic spacing or bond length r_o . In the case of Young's elastic modulus E , we define it in terms of bond length r and internal energy U such that

$$E = \left. \frac{\partial^2 U / \partial r^2}{r_o} \right|_{r_o} = \frac{\text{Newtons}}{\text{meters}^2} = \frac{k}{r_o}.$$

Thus, E is bond stiffness normalized by bond length.

- (b) Now provide the approximate derivation of the following elastic constants in terms of thermodynamic state variables U and S , and the applied strain ϵ_{ij} : K , the bulk elastic modulus indicating resistance to dilatation (volumetric strain change) under an applied stress $\sigma_{ij} = \sigma_{ii}$; and E , the Young's elastic modulus. Give your solution in terms of $K(U, S, V, \epsilon)$ and $E(U, S, V, \epsilon)$, where V is the unit volume of material.

Solution: Under volumetric strain $\epsilon_{xx} = \epsilon_{yy} = \epsilon_{zz} = \epsilon_o$, an elastomer expands uniformly without changing configuration among networked chains or in overall shape. Therefore

$$K \sim \frac{1}{V} \frac{d^2 U}{d\epsilon^2}$$

Under uniaxial loading, large configurational changes occur as molecular chains stretch and align, resulting in a mostly entropic contribution. Therefore

$$E \sim -\frac{T}{V} \frac{d^2 S}{d\epsilon^2}$$

- (c) For an elastomer, state whether you expect K to be large or small in magnitude; and E to be large or small in magnitude.

Solution: K would be large (resistant to dilation or expansion/contraction under hydrostatic stress) and E would be low (not resistant to uniaxial extension as compared to metals).

- (d) The Poisson's ratio of a material can be expressed in terms of K , E and G . For example,

$$\nu = \frac{1}{2} - \frac{1}{6} \frac{E}{K}$$

Based on this relation, what is the expected lateral strain resulting from an axial compressive strain of 100% imposed on an elastomer?

Solution: Because we expect large K and small E , we can say that E/K tends to zero. Therefore $\nu = 1/2$ and the material will be 50% wider if it becomes 100% shorter.

- (e) Besides the cis-isoprene latex example discussed in class, give the *name* of at least one other actual material that is expected to exhibit elastomeric behavior at room temperature of 22°C.

Solution: The purpose of this question is to confirm that you know which actual materials can be analyzed as elastomers, and to confirm a difference between a single chain capable of large reversible extensions (e.g., WLC) and an elastomeric network that can be used in engineering applications. For an elastomer, the material must have kinked chains that confer an entropic penalty upon unkinking (which, individually, could be WLCs), a network with crosslinks to establish a restoring force upon removal of stress, and a glass transition temperature less than room temperature to confer sufficient molecular mobility. An example of such an elastomer is PDMS, or poly(dimethylsiloxane), which is basically silly putty.