Solving Fluid Dynamics Problems

3.185

November 29, 1999, revised October 31, 2001, November 1, 2002, and November 5, 2003

This outlines the methodology for solving fluid dynamics problems as presented in this class, from start to finish. ("W^{3}R" references are to the textbook for this class by Welty, Wicks, Wilson and Rorrer.)

1. Set up the problem

    (a) What is the question you are trying to answer?

    (b) Sketch the geometry

    (c) Choose the coordinate system, to determine which form of the Navier-Stokes equations will be the simplest to use

    (d) Label the coordinates of problem features

2. Choose the form of the Navier-Stokes equations, and simplify using assumptions

    (a) Is the fluid Newtonian?

        i. The following fluids are generally *not* Newtonian:
            A. liquid polymers, including liquid crystals (these may also be anisotropic)
            B. gels
            C. solid-liquid mixtures with high solid fraction, such as semisolid metals and ceramic slurries
        ii. If Newtonian, use the stress given by:

$$\tau = -\mu \left(\nabla \vec{u} + (\nabla \vec{u})^T - \frac{2}{3} \nabla \cdot \vec{u} I \right).$$

        iii. Is its viscosity constant?
            A. Viscosity is generally a function of temperature and composition, if both of these are both constant it's a good bet viscosity is too.
            B. If constant, you can pull the viscosity μ out of shear stress derivatives in the momentum equation.

    (b) Is the fluid compressible?

        i. Most liquids can be treated as incompressible for this class.
        ii. Gases can also be treated this way if the mach number (maximum velocity divided by the speed of sound) is below about 0.3 and pressure does not vary by more than about 5%.
        iii. If incompressible, $\frac{D\rho}{Dt} = 0$, so the continuity equation reduces to $\nabla \cdot \vec{u} = 0$.
        iv. Is the density constant?
            A. For any single liquid, probably yes; for a multi-phase liquid mixture (*e.g.* oil/water), probably no.
            B. If constant, you can take ρ out of derivatives in the continuity equation.

C. If the fluid is also Newtonian with constant viscosity, you can use the components of the momentum equation given in the Navier-Stokes equations in W^{3}R Appendix E (pp. 704–705). Otherwise you must use the more general equations given in vector form below and substitute your constitutive equation for the shear stress.

(c) Is the flow laminar?

    i. The Reynolds number Re=$\rho u L/\mu$ will largely determine this.

      A. For pipe flow, use the diameter for L, if Re<~2000, it will be laminar.

      B. For a free-falling liquid film, use the film thickness for L, the threshold is about 20.

      C. For a growing boundary layer, use distance from the leading edge x instead of L to give the local Reynolds number Re$_x$. The boundary layer will be laminar until Re$_x = 10^5$, transitional for $10^5 <$ Re$_x < 3 \times 10^6$, and turbulent for Re$_x > 3 \times 10^6$.

    If these thresholds are exceeded then the fluid may be transitional or fully turbulent depending on entrance conditions, ambient noise, etc.

    ii. If laminar, use the Navier-Stokes equations as they are.

    iii. If turbulent, you will need to solve the time-smoothed Navier-Stokes equations with a turbulence model to achieve closure on the Reynolds stresses (or else use a supercomputer to solve all of the velocity and pressure fluctuations at all length scales down to the smallest eddies).

(d) Is the flow time-dependent or steady-state?

    i. Think about whether the velocity is changing at any given point in space in a fixed frame of reference. If so, it is time-dependent, if not, it is steady-state.

    ii. If steady-state, you can ignore all time derivatives (NOT the substantial derivatives, just the partial derivatives with respect to time).

(e) Is the flow fully-developed?*

    i. Free flows (boundary layers, jets, flow past a sphere, etc.) are *never* fully-developed.

    ii. For confined flows, if the entrance length is a small fraction of the overall length, and the question posed does not focus on what happens at the entrance, then you can make this assumption.

    iii. If fully-developed, the flow is roughly unidirectional, so you can ignore components of velocity not in that direction (if the transverse flow drivers are uniform). You can also ignore velocity derivatives along the flow direction.

(f) Can you neglect edge effects?*

    i. If the geometry is much wider than it is thick, the driving force for flow is uniform across the width, and the problem does not depend on what happens at the edges, then you can make this assumption.

    ii. If so, you can ignore lateral velocity derivatives (in the width direction).

3. Solve the resulting simplified equations by integration to give the pressure and velocity fields

    (a) To solve for the pressure distribution in fully-developed flow, it is sometimes helpful to differentiate the equation in the flow direction, giving an expression for the second derivative of pressure in that direction.

4. Determine the boundary conditions

    (a) If the boundary is a solid:

      i. use the no-slip boundary condition, *i.e.* the fluid velocity equals the solid velocity

    (b) If the boundary is a free surface:

      i. the velocity normal to the surface $\vec{u} \cdot \hat{n}$ is equal to the rate of motion of the surface (zero for steady-state)

      ii. the shear on that surface in the tangent directions is zero (or equal to the surface tension gradient, but that's beyond 3.185)

      iii. The pressure equals the atmospheric pressure (plus the product of surface tension and mean curvature, but that's beyond 3.185)

  (c) For a symmetry plane or axis of symmetry:

      i. the normal velocity to the plane is zero (if axisymmetric, the r- and θ-velocities are zero on the axis)

      ii. the shear on that surface/axis τ_{zr} and τ_{rz} is zero

5. Use the boundary conditions to determine the integration constants to derive an expression for the flow velocity and pressure fields

6. To calculate volume flow rate Q through a surface S (typically a cross section of the flow), use the integral

$$Q = \int\int_S \vec{u} \cdot \hat{n} \, dA$$

To calculate mass flow rate $\dot{M}$ through a surface S, use

$$\dot{M} = \int\int_S \rho \vec{u} \cdot \hat{n} \, dA$$

7. To calculate the average velocity, divide the flow rate by cross section area

8. To calculate shear drag force on a surface, integrate the shear stress over that surface

  (a) Shear stress can be obtained from the velocity field using the constitutive equation.

  (b) For many situations, shear stress will be constant, so just multiply by area.

9. To calculate torque around an axis exerted on a surface, integrate the product of tangential shear stress and radius $r\tau_{r\theta}$ over the surface.

* These assumptions will give you incorrect velocity and pressure values in the entrance and edges respectively, but for purposes of calculating bulk flow properties, such flow rate or total shear drag force, these errors should be very small.

The Navier-Stokes Equations in Vector Form

The continuity equation (conservation of mass):

$$\frac{D\rho}{Dt} + \rho \nabla \cdot \vec{u} = 0$$

The motion equation (conservation of momentum):

$$\rho \frac{D\vec{u}}{Dt} = -\nabla p - \nabla \cdot \tau + \rho \vec{g}$$

The same with Reynolds stresses:

$$\rho \frac{D\vec{u}}{Dt} = -\nabla p - \nabla \cdot (\tau_l + \tau_t) + \rho \vec{g}$$

where τ_t is the reynolds stress tensor, $\tau_t = \rho \vec{\bar{u}'} \vec{u}'$.

The motion equation for an incompressible fluid with constant viscosity:

$$\rho \frac{D\vec{u}}{Dt} = -\nabla p + \mu \nabla^2 \vec{u} + \rho \vec{g}$$

For the Navier-Stokes equations in Cartesian and spherical coordinates, see W^{3}R Appendix E, pp. 704-705.

Cylindrical Coordinates For a Newtonian incompressible fluid in cylindrical coordinates (W^{3}R pp. 704-705):

$$\text{mass}: \quad \frac{1}{r}\frac{\partial}{\partial r}\left(ru_r\right) + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z} = 0 \tag{1}$$

$$r-\text{momentum}: \quad \rho\left(\frac{\partial u_r}{\partial t} + u_r\frac{\partial u_r}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta^2}{r} + u_z\frac{\partial u_r}{\partial z}\right) = \tag{2}$$

$$-\frac{\partial p}{\partial r} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(ru_r\right)\right) + \frac{1}{r^2}\frac{\partial^2 u_r}{\partial \theta^2} - \frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial^2 u_r}{\partial z^2}\right] + F_r$$

$$\theta-\text{momentum}: \quad \rho\left(\frac{\partial u_\theta}{\partial t} + u_r\frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r u_\theta}{r} + u_z\frac{\partial u_\theta}{\partial z}\right) = \tag{3}$$

$$-\frac{1}{r}\frac{\partial p}{\partial \theta} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(ru_\theta\right)\right) + \frac{1}{r^2}\frac{\partial^2 u_\theta}{\partial \theta^2} + \frac{2}{r^2}\frac{\partial u_r}{\partial \theta} + \frac{\partial^2 u_\theta}{\partial z^2}\right] + F_\theta$$

$$z-\text{momentum}: \quad \rho\left(\frac{\partial u_z}{\partial t} + u_r\frac{\partial u_z}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_z}{\partial \theta} + u_z\frac{\partial u_z}{\partial z}\right) = \tag{4}$$

$$-\frac{\partial p}{\partial z} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u_z}{\partial \theta^2} + \frac{\partial^2 u_z}{\partial z^2}\right] + F_z$$

For a non-Newtonian or compressible fluid in cylindrical coordinates:

$$\text{mass}: \quad \frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}\left(\rho r u_r\right) + \frac{1}{r}\frac{\partial}{\partial \theta}\left(\rho u_\theta\right) + \frac{\partial}{\partial z}\left(\rho u_z\right) = 0 \tag{5}$$

$$r-\text{momentum}: \quad \rho\left(\frac{\partial u_r}{\partial t} + u_r\frac{\partial u_r}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta^2}{r} + u_z\frac{\partial u_r}{\partial z}\right) = \tag{6}$$

$$-\frac{\partial p}{\partial r} - \left(\frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rr}) + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} - \frac{\tau_{\theta\theta}}{r} + \frac{\partial \tau_{rz}}{\partial z}\right) + F_r$$

$$\theta-\text{momentum}: \quad \rho\left(\frac{\partial u_\theta}{\partial t} + u_r\frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r u_\theta}{r} + u_z\frac{\partial u_\theta}{\partial z}\right) = \tag{7}$$

$$-\frac{1}{r}\frac{\partial p}{\partial \theta} - \left(\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\tau_{r\theta}) + \frac{1}{r}\frac{\partial \tau_{\theta\theta}}{\partial \theta} + \frac{\partial \tau_{\theta z}}{\partial z}\right) + F_\theta$$

$$z-\text{momentum}: \quad \rho\left(\frac{\partial u_z}{\partial t} + u_r\frac{\partial u_z}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_z}{\partial \theta} + u_z\frac{\partial u_z}{\partial z}\right) = \tag{8}$$

$$-\frac{\partial p}{\partial z} - \left(\frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rz}) + \frac{1}{r}\frac{\partial \tau_{\theta z}}{\partial \theta} + \frac{\partial \tau_{zz}}{\partial z}\right) + F_z$$