3.185 - Recitation Notes

September 04/05, 2003

Topics Covered:

- Dot Product
- Cross Product
- Outer Product
- Gradient of Scalar Field
- Divergence of a Vector Field
- Curl of a Vector Field
- Ordinary Differential Equation
- Error Function
- Substantial Derivative of Scalar Field

Dot Product

$$u = \begin{pmatrix} u_1 & u_2 & u_3 \end{pmatrix}$$
$$v = \begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix}$$
$$u \cdot v = u_1 v_1 + u_2 v_2 + u_3 v_3$$

Cross Product

$$u \times v = \begin{vmatrix} i & j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$
$$u \times v = (u_2 v_3 - u_3 v_2)i + (u_3 v_1 - u_1 v_3)j + (u_1 v_2 - u_2 v_1)k$$

Outer Product

$$u \otimes v = \begin{pmatrix} u_1 v_1 & u_1 v_2 & u_1 v_3 \\ u_2 v_1 & u_2 v_2 & u_2 v_3 \\ u_3 v_1 & u_3 v_2 & u_3 v_3 \end{pmatrix}$$

Del Operator

$$\nabla = i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}$$

Gradient of Scalar Field

Suppose Ø(x,y,z) is a scalar field

$$\text{Grad } \varnothing(x,y,z) = \nabla\phi(x,y,z) = \frac{\partial\phi}{\partial x}i + \frac{\partial\phi}{\partial y}j + \frac{\partial\phi}{\partial z}k$$

Divergence of a Vector Field

Suppose **F**(x,y,z) is a vector field: **F**= F_1(x,y,z)i + F_2(x,y,z)j + F_3(x,y,z)k

$$\text{Div. } \mathbf{F}(x,y,z) = \nabla\cdot\vec{F}(x,y,z) = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = \text{A scalar}$$

Interpretation of the *divergence*: "… a measure of the rate at which the field 'diverges' or 'spread away' from a point".

Curl of a Vector Field

Suppose **F**(x,y,z) is a vector field: **F**= F_1(x,y,z)i + F_2(x,y,z)j + F_3(x,y,z)k

$$\text{Curl } \mathbf{F}(x,y,z) = \nabla\times\vec{F}(x,y,z) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$\nabla\times\vec{F}(x,y,z) = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)i + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)j + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)k = \text{A vector}$$

Interpretation of the *curl*: "… measures the extent to which the vector field **F** 'swirls' around a point".

Ordinary Differential Equation

- 1^{st} order

  Ex.: $\frac{dy}{dx} = ky$ … the solution is $y = ce^{kx}$

- 2^{nd} order

  Ex.: $\frac{d^2y}{dx^2} - y = 0$

  By inspection, one of the solutions is $y = ce^{x}$, but wait… $y = ce^{-x}$ is also a solution In fact, the linear combination of any individual solution is also a solution; therefore, $y = c_1e^{kx} + c_2e^{-kx}$ is also a solution. Boundary conditions are required to identify the particular solution from the family of solutions.

  For more general 2^{nd} order ODE, we can write $ay'' + by' + cy = g(x)$.
  When g(x) = 0, the ODE is said to be homogeneous - $ay'' + by' + cy = 0$

  Let's suppose the solution to the homogeneous 2^{nd} order ODE is: $y = e^{rx}$

$$y' = re^{rx}$$
$$y'' = r^2e^{rx}$$
$$ay'' + by' + cy = 0$$
$$(ar^2 + br + c)e^{rx} = 0$$
$$\text{since } e^{rx} \neq 0$$
$$ar^2 + br + c = 0$$

(this is called the characteristic equation)

Note: The quadratic equation has two roots, which can be real, complex or repeated.

The complete solution to $ay'' + by' + cy = 0$ is

$$y = c_1 e^{r_1 x} + c_2 e^{r_2 x}$$

(This can be verified easily by back substitution)

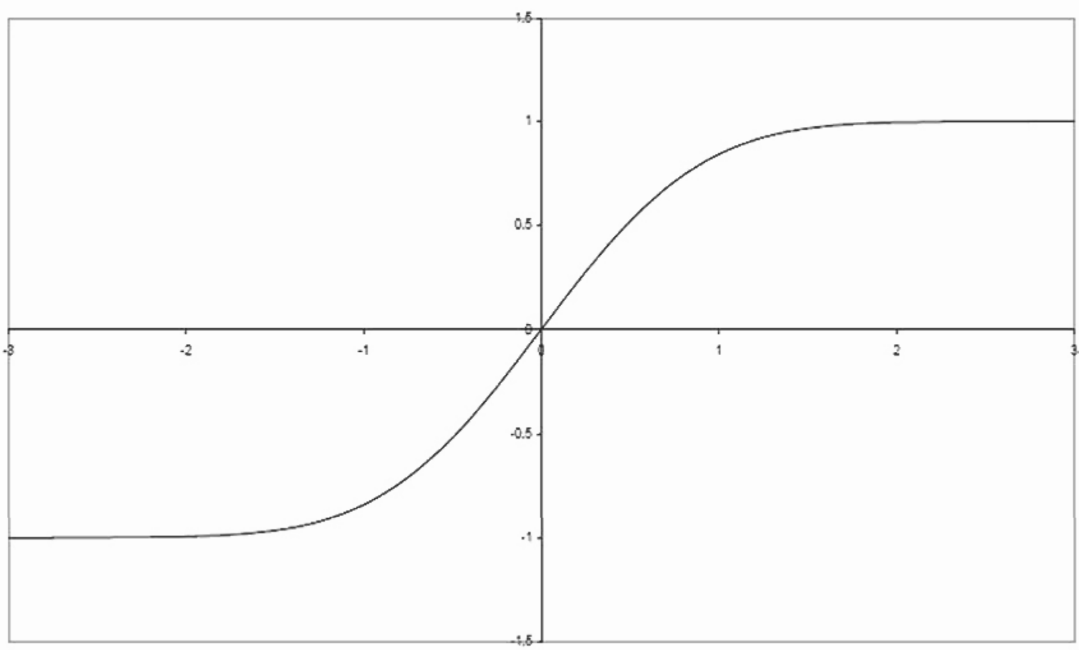
<u>Error Function</u>

Definition:

$$erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-\zeta^2} d\zeta$$

Some properties:

$$erf(0) = 0$$
$$erf(\infty) = 1$$
$$erf(-x) = -erf(x)$$



<u>Substantial Derivative</u>

Definition:

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + v_x \frac{\partial}{\partial x} + v_y \frac{\partial}{\partial y} + v_z \frac{\partial}{\partial z}$$

$$v_x = \frac{dx}{dt}$$

where $v_y = \frac{dy}{dt}$

$$v_z = \frac{dz}{dt}$$

$$\frac{D}{Dt} = \underbrace{\frac{\partial}{\partial t}}_{\substack{\text{local}\\ \text{rate}\\ \text{of}\\ \text{change}}} + \underbrace{v_x \frac{\partial}{\partial x} + v_y \frac{\partial}{\partial y} + v_z \frac{\partial}{\partial z}}_{\text{rate of change due to motion}}$$

Also read W^3R p.111

Reminder: Thursday recitation time change: 1:00am – 1:00pm?