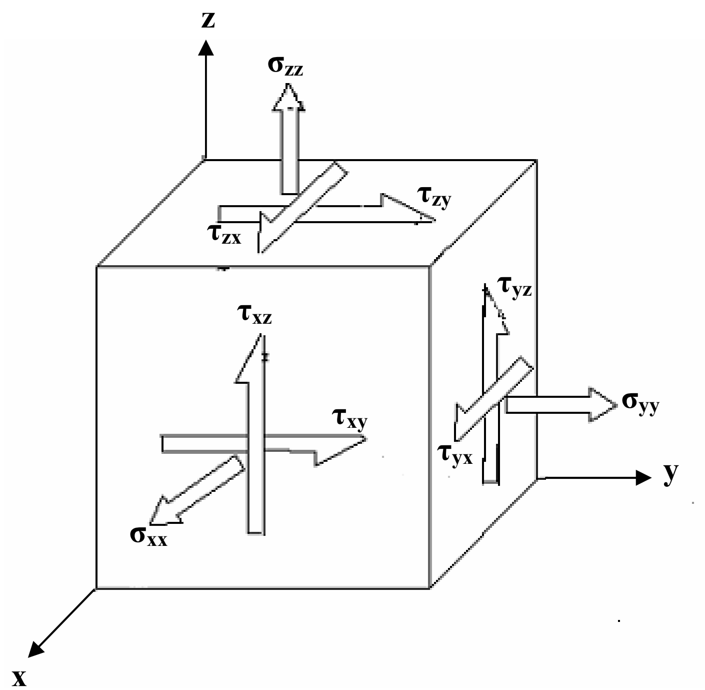
3.032 Problem Set 3 Solutions

Fall 2007

Due: **<u>Start</u>** of Lecture, 10.01.07

1. Here, we will analyze the stress states of a solid block of aluminum under various loading conditions using the Mohr's Circle construction. We are going to use the following convention to describe the different components of stress:



For each of the following situations, describe the stress state in terms of a stress matrix σ_{ij}. Determine the principal normal and shear stresses and give the orientation of the principal axes as well as the orientation of the max shear stress.

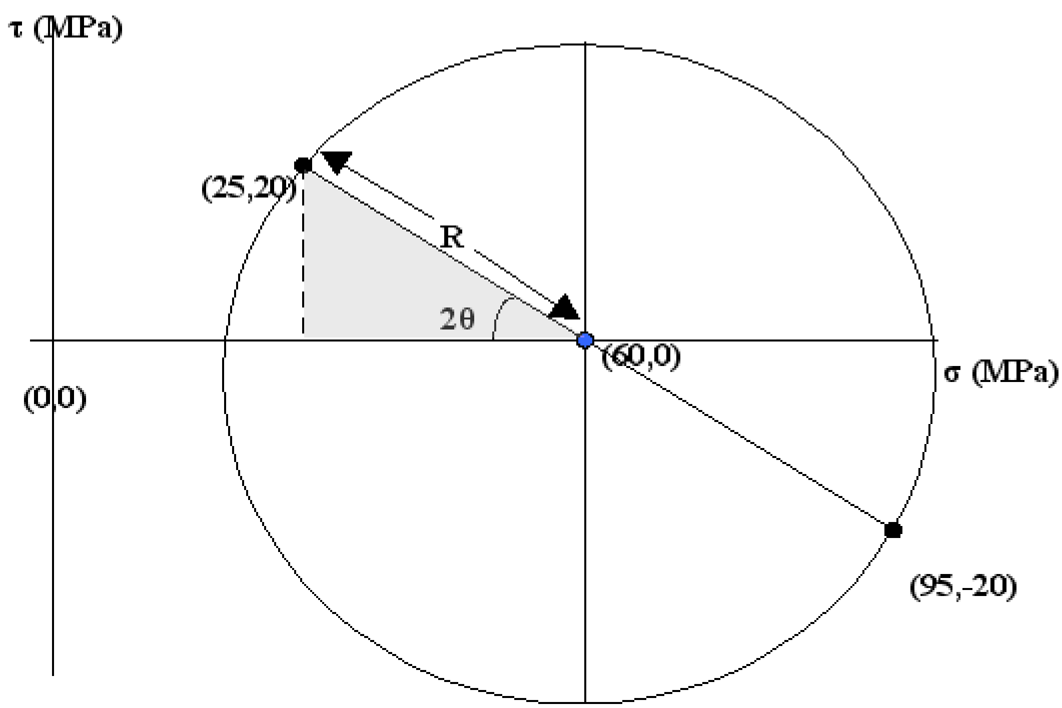
(a) One very popular application of aluminum is for use as thin films for metallization layers in integrated circuit (IC) fabrication. In the case of thin-films (or other thin geometries), where one of the dimensions of the material is much smaller than the other two, the stress acting through the smaller dimension is negligible and can be neglected during analysis. This case, where the stresses acting on one of the orthogonal planes is zero, is known as *plane stress*. Let us assume that the thickness dimension of the Al thin-film is along the z-direction (parallel to the x-y plane), so that the plane stress condition is $\sigma_{zz} = \tau_{xz} = \tau_{yz} = 0$.

During a bend test of the Al thin film, a point on the surface of the film was found to have high stresses. It was decided that this point of the thin film would be further analyzed. The stresses on the film, with respect to the coordinate system shown above, were found to be σ_{xx} = 95MPa, σ_{yy} = 25MPa, and τ_{xy} = 20MPa. Determine all of the information stated in the problem (in bold italics) and note the value of the maximum normal stress.

Solution: For this plane stress situation, there is a two-dimensional state of stress which can be described using the following stress matrix

$$\sigma_{ij} = \begin{bmatrix} 95 & 20 & 0 \\ 20 & 25 & 0 \\ 0 & 0 & 0 \end{bmatrix} MPa$$

To obtain the Mohr's Circle for this stress state, we plot the two points that lie at opposite ends the diameter (solid black dots). The two points are (σ_x, -τ_{xy}) and (σ_y, τ_{yx}). The reason for the positive/negative signs is due to our chose of convention for the stresses. A positive τ_{xy} is seen to produce a counter-clockwise rotation (hence, it is negative in out Mohr's Circle construction) and a positive τ_{yx} is seen to produce a clockwise rotation (hence, it is positive in our Mohr's Circle construction). The grouping of the points is chosen so that all the stresses are acting on the same plane (or face) of the material.



The center of the circle is located at the average normal stress value which is calculated as followed

$$\sigma_{avg} = \frac{\sigma_x + \sigma_y}{2} = \frac{95 + 25}{2} = 60MPa$$

Thus, the center of the circle is located at (60,0) MPa, as noted by the blue dot.

Next, we calculate the radius of the circle using the following equation

$$R = \sqrt{\left(\frac{\sigma_{11} - \sigma_{22}}{2}\right)^2 + \sigma_{12}{}^2} = 40.3MPa$$

(Note: This is the same as calculating the hypotenuse of the right triangle shaded in grey.)

The principal normal stresses σ_1 and σ_2 occur at an orientation defined by the angle 2θ between the original plane and the line at $\tau = 0$; and the maximum shear stress occurs at an orientation that is defined by the angle 2θ between the original plane and the vertical diameter at $\sigma = 60$ MPa . The magnitudes are calculated as follows:

$$\sigma_1 = \sigma_{avg} + R = 60 + 40.3 = 100.3MPa$$
$$\sigma_2 = \sigma_{avg} - R = 60 - 40.3 = 19.7MPa$$
$$\tau_{12} = R = 40.3MPa$$

Thus, the maximum normal stress is 100.3MPa. The orientation of the principal normal stresses is at an angle of 2θ from the σ-axis. This angle can be calculated as follows

$$\tan(2\theta) = \frac{20}{35}$$
$$2\theta = 29.74^\circ$$

Therefore, a counterclockwise rotation of the diameter of the circle by 29.74° (or a 14.9° rotation in the same direction about the z-axis in the material) will produce the principal stress state

$$\sigma_{ij} = \begin{bmatrix} 100.3 & 0 \\ 0 & 19.7 \end{bmatrix} MPa$$

The orientation of maximum shear stress is given by a clockwise rotation of the diameter by 60.26° (or a 30.13° rotation in the same direction about the z-axis in the material). The stress state in this case can be written as

$$\sigma_{ij} = \begin{bmatrix} 60 & 40.3 \\ 40.3 & 60 \end{bmatrix} MPa$$

(b) Aluminum is also used in electrical transmission lines for power distribution. In order to assess the mechanical integrity of the lines, the lines are loaded in uniaxial tension. It is observed that the lines permanently deform at an applied uniaxial stress of 20 MPa. In addition to the requirements in bold font (**σ_{ij}**, magnitude and orientation of principal/max shear stress states), also **draw** the orientation of the material representative volume element (here, a plane) that is under maximum shear stress. Assume the following orientation and axis-set for the Al lines loaded in tension:

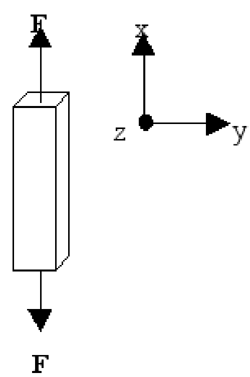
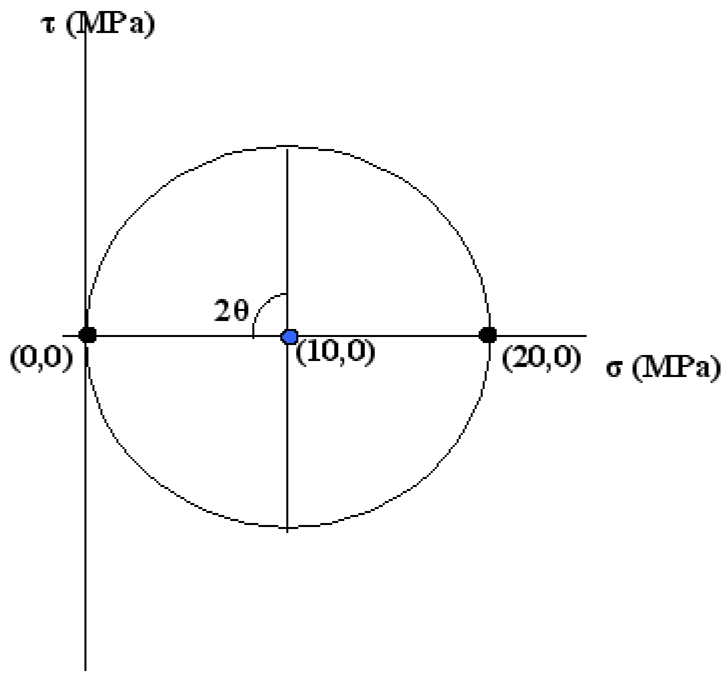


Fig. 1a: Aluminum power line loaded under uniaxial force F.

Solution: In this case we only have one normal stress acting in the x-direction and no shear stresses. Thus, the stress state is given as

$$\sigma_{ij} = \begin{bmatrix} 20 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} MPa$$

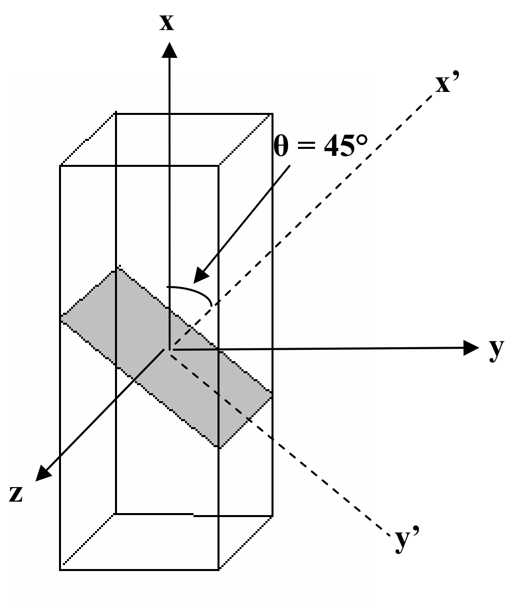
The Mohr Circle construction is quite simple in this case. The two endpoints of the radius will be at (0,0) and (20,0) and the circle is drawn as shown below



We see by simple visual inspection of the Mohr circle that we are already in the principal stress state (i.e. all shear stresses are equal to zero), so we do not need to rotate our circle (or material) any further. The principal axis is the x-axis. Also, there is only one principal normal stress which is equal to 20MPa.

In this case, the radius of the circle is equal to the average stress which is 10MPa [(20+0)/2]. This is also the maximum shear stress. The diameter corresponding to the original state of stress must be rotated clockwise an angle of 2θ equal to 90° to obtain the equivalent representation that contains the principal shear stress. This is the same as rotating the material in the same direction an angle of 45°.

An illustration of the plane of max shear stress is shown below (shaded in grey). Notice that the plane of max shear stress is oriented 45° from the principal axis, which is the direction of the applied normal force in this case. Rotating the x-y plane clockwise 45° about the z-axis gives the equivalent representation that contains the principal shear stress



(c) One common forming process for aluminum parts is known as *extrusion*. In the extrusion process, a billet of material is forced through a die in order to obtain a workpiece with the desired cross-sectional geometry. One type of extrusion process is known as *hydrostatic extrusion*. Unlike many extrusion operations, where pressure to the billet is supplied by a hydraulically-driven ram or pressing stem, the pressure in hydrostatic extrusion is supplied through an incompressible fluid medium surrounding the billet (see Figure 1b below). Hydrostatic extrusion is a very popular operation for ductile materials such as aluminum due to its ability to reduce defects in the extruded part through the compressive environment. Typical pressures exerted on the workpiece are around 1400MPa.

Write the stress state of a representative volume element of the Al billet inside of the extruder under the hydrostatic pressure of 1400 MPa (noted by the grid region in Fig. 1b), and determine all other information requested in bold font.

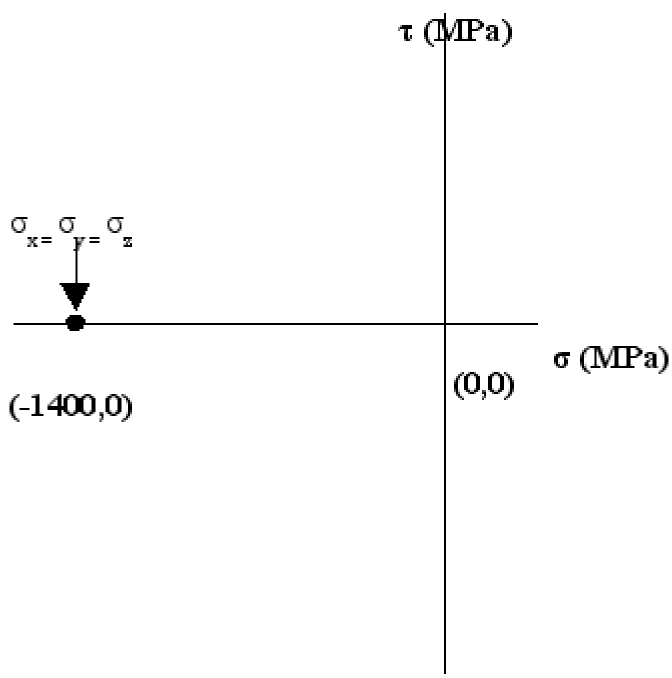
Figure 1b: Hydrostatic extrusion process

(www.fzs.tu-berlin.de/html/en/strpr_hydro.html)

Solution: For the hydrostatic pressure condition, the normal stresses acting on all three pairs of orthogonal faces of the block of material are equal. Also, all of the shear stresses are equal to zero. Since the pressure being exerted on the Al billet is compressive, the normal stresses will all have negative values. Thus, the stress state will look like

$$\sigma_{ij} = \begin{bmatrix} -1400 & 0 & 0 \\ 0 & -1400 & 0 \\ 0 & 0 & -1400 \end{bmatrix} MPa$$

The Mohr's Circle construction for this stress state will just be a single point because all of the normal stresses are equal and there are no shear forces. This is shown in the diagram below



As all of the shear forces are equal to zero, we see that we are already in the principal stress state with each of the principal normal stress being equal to -1400MPa. The principal axes are the x-, y-, and z-axes as noted in our original diagram and no rotation is necessary.

One very interesting and important observation to take away is that the maximum shear stress, τ_{max}, is zero (R = 0) under hydrostatic pressure! Implications of this fact will be discussed further in class.

2. You are responsible for performing uniaxial tensile tests on three very different materials: a 316 stainless steel alloy, alumina (Al_2O_3), and high density polyethylene (HDPE). However, before performing the actual tests, you are asked to predict what the **elastic** stress vs. strain responses of each of the materials based on the mechanical properties of these materials documented in the literature (e.g., material property databases such as matweb.com, linked on our MIT Server site).

(a) Graph the stress [MPa] versus strain [%] response for all three materials on a single graph and on 3 separate graphs, up to an applied strain value of .01 (or 1%) in strain increments of 0.0005.

(b) Remark on the differences in behaviors seen for each of the three materials as related to their relevant mechanical properties. Also, looking at the magnitude of the stresses at the maximum applied strain, do you expect all of these materials to deform elastically up to these strains? If not, what was the fallacy in ***solely*** using Hooke's law to predict the stress-strain behaviors for each of the materials?

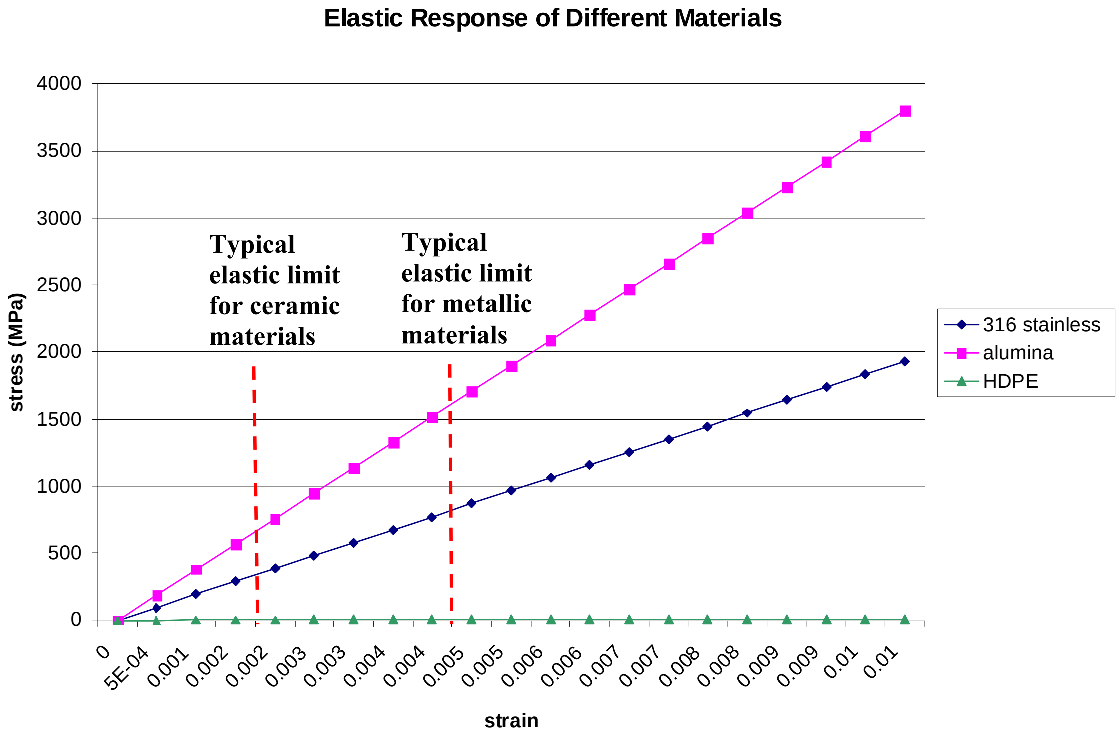
Solution: The Young's modulus for each of the three materials (as found in Callister, 'Materials Science and Engineering An Introduction', 5th ed.) are

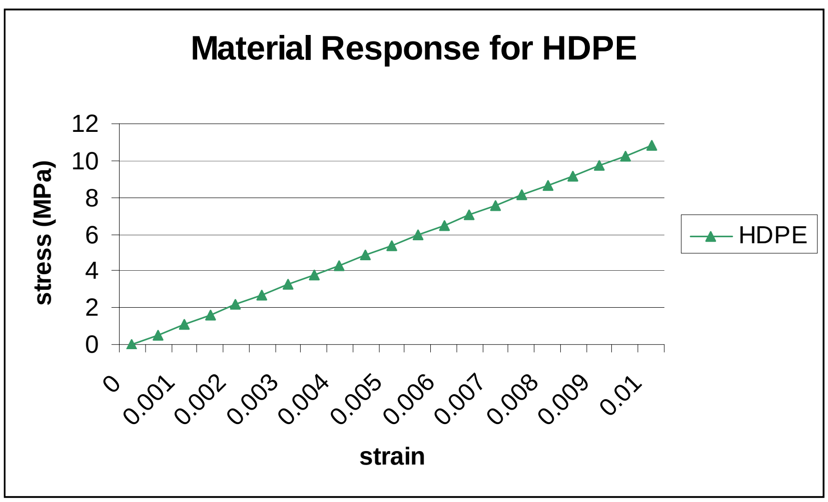
E (316 stainless steel) = 193 GPa
E (alumina) = 380 GPa
E (HDPE) = 1.08 GPa

The general formula for linearized Hooke's law is given by

$$\sigma = E\varepsilon$$

Using the respective value of Young's modulus for each of the three materials, the stress-strain graphs can be plotted by using Hooke's law.





One can observe that the plots for all three materials produce straight lines with the slope being equal to the Young's Modulus of the material. The plot for HDPE was reproduced on a second graph to make the stress values readable.

Comparing the graphs of the different materials it is seen that alumina has the largest slope, the 316 stainless steel has next largest, and finally HDPE has the smallest slope. This makes sense based on the relative values of their Young's modulus. The physical meaning behind this difference is the resistance of the material to elastic deformation. The Young's modulus is a measure of the material's stiffness, and hence its resistance to elastic deformation. Thus, the higher the Young's modulus, the more stress is required to produce the same amount of elastic strain. Ceramic materials, such as alumina, tend to be very stiff and thus have high values of E. On the other hand, polymers, such as HDPE, are very elastic and thus *tend to have very small values of E. Metals, such as the stainless steel, usually have modulus values between the two types.*

If we take a look at the curves for the alumina (a ceramic) and the 316 stainless steel (a metal), we see that the stress reaches magnitudes of 3800MPa and 1930MPa, respectively, at a strain of .01 (or 1%). These stresses are extremely high and most materials would either yield (typical metals) or fracture (typical ceramics) under such stresses. In fact, for most metallic materials, elastic deformation persists only to strains of about .005 (5%), after which plastic deformation processes start to take place. Ceramics, being very brittle materials, undergo even less elastic deformation and tend to plastically deform at strains of around .002 (2%) or less. Both of these limits are shown on the graph above. (Note: These are just average values and the elastic limits, and corresponding stress values, will vary depending on the specific material).

Unlike most metals and ceramics, polymers can undergo very large strains without undergoing plastic deformation. However, Hooke's law only accurately describes the stress-strain response for polymers at low temperatures for relatively small strains. At higher strains and temperatures, other non-linear processes govern deformation.
*To summarize, the most important note to take away from this problem is that Hooke's Law is only valid in the **elastic** region of the stress-strain behavior. Past this region, different plastic deformation processes take place and result in much different (non-linear) stress-strain behaviors. These plastic deformation processes differ for different material classes, and many of them will be explored throughout this class!*

(c) All three samples were given to you as cylinders with identical initial dimensions of 10 cm length and 2 cm diameter. Show whether a uniaxial load frame of maximum load capability = 100 kN (like the ones you used to crush the beverage cans in Lab 1) will be sufficient to deform all three materials to the requested engineering normal strain of 1%. Here, neglect the possibility that the materials might not remain intact to that strain.

*Solution: In order to determine the force necessary to deform the materials to a strain of 1%, we must first find the associated stress necessary to obtain that same strain. We can use Hooke's law (assuming all the materials behave in a **linear elastic** manner up to that strain) to figure out the stress needed to deform the materials to a 1% strain:*

$$\sigma_{steel} = (193\times10^{9})(.01) = 1930MPa$$

$$\sigma_{Al_2O_3} = (380\times10^{9})(.01) = 3800MPa$$

$$\sigma_{HDPE} = (1.08\times10^{9})(.01) = 10.8MPa$$

Next, we can calculate the cross-sectional area for the cylindrical specimens

$$A = \pi r^2 = \pi(1\times10^{-2})^2 = 3.14\times10^{-4}\,m^2$$

Finally, we calculate the force required to produce the 1% strain by using our definition of engineering stress for the axial loading condition ($\sigma=F/A_o$ or $F=\sigma A_o$)

$$F_{steel} = (1930\times10^6)(3.14\times10^{-4}) = 606.0kN$$

$$F_{Al_2O_3} = (3800\times10^6)(3.14\times10^{-4}) = 1193.2kN$$

$$F_{steel} = (10.8\times10^6)(3.14\times10^{-4}) = 3.4kN$$

Therefore, the axial load frame will only be able to deform the HDPE to a 1% strain as the force required to deform the steel and alumina to the same strain are greater than the maximum load capacity of 100kN.

3. You and your labmate have been given a joint project by your UROP advisor. He has asked you both to measure all the elements of the stiffness tensor $\mathbf{C}_{ijkl}$ of two new proteins he has crystallized and is considering for use in a flexible, organic integrated circuit that must withstand mechanical bending.

Another student has already used x-ray diffraction to determine the crystal structure of the proteins: protein A exhibits a tetragonal unit cell, whereas protein B exhibits an orthorhombic unit cell. Your labmate says, "Let's each measure the elastic constants $\mathbf{C}_{ijkl}$ of one protein. Which one do you want to analyze, protein A or B?'"

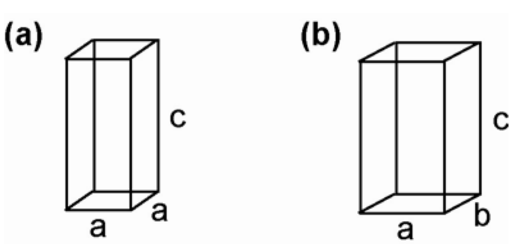


Fig. 3: Unit cells for (a) tetragonal and (b) orthorhombic crystals.

Under the assumption that you'd like to spend as little time as possible on this project so you can get back to your 3.032 studies, what is your answer? Explain concisely, but as fully and accurately as possible, using the elasticity concepts discussed in 3.032.

Solution:
These protein crystals are structurally anisotropic, so the stiffness tensor $\boldsymbol{C}_{ijkl}$ could contain up to 81 components (a 9 x 9 matrix mapping the 3x3 stress and strain tensors to each other), at least 21 of which could be independent values. This means I could need to make as many as 21 different tests, loading the crystal in 21 different uniaxial directions.

However, I learned in 3.032 that this number of independent elastic constants decreases as the symmetry of the material (crystal structure, microstructure, or macrostructure as in a composite) increases. For an isotropic linear elastic material, there are only 2 independent elastic constants, so I'd like to choose the protein crystal that is most like an isotropic material (elastic constants independent of measurement direction).

Here, the tetragonal crystal is more symmetric than the orthorhombic crystal, which is clear from the fact that two of the lattice parameters of the tetragonal unit cell are indistinguishable. I'll choose to measure the elastic constants for protein A.

How many are there? Tetragonal has 7 independent elastic constants, and orthorhombic (a less symmetric unit cell) has 9. You were not asked to state this for pset 3, but you can figure it out from transformation of crystallographic axis directions that are equivalent. For the tetragonal crystal, if we assume the 3-direction is in the c-axis or long direction, the stiffness in the 1- and 2-directions must be equivalent, so $C_{1111} = C_{2222}$ *or* $C_{11} = C_{22}$ *in contracted notation of the 6 x 6 matrix* C_{ij}*. Likewise, the shear moduli in those planes would be equivalent, so C44 = C55. And so on. Ultimately, tetragonal crystals have at most 7 independent elastic constants: 11, 12, 13, 16, 33, 44, and 66.*